Math 4500/6500 Exam #1

This take-home exam covers the material from Chapter 1 through 4 of Cheney/Kincaid, from floating point numbers through Polynomial Interpolation. The exam is intended to be done with the aid of a computer program such as Mathematica.

Please don't be afraid to read over the notes and these sections in the book as you work on the problems– there is more in the notes than we were able to cover in the lectures, and some of those extra facts might be helpful to you as you work on the exam problems.

Pick 5 of the following problems. 6500 students must include problem 4. All students must include a computer problem in their exam. You are encouraged to use the computer on as many problems as you like. It is ok to use Mathematica to do algebra for you, for instance.

You **are** permitted to use	You **are not** permitted to use
Maple (or Mathematica or MATLAB)	The internet (except for Mathematica help)
Our book	Other books
Your notes	Other people's notes
Your brain	Other people's brains
Class notes posted on the website	
Mathematica code posted on the website	

1. The secant method for finding the root of a function $f(x)$ uses two successive points x_n, x_{n-1} to compute a projected root based on the secant line through x_{n-1} and x_n. Using your knowledge of polynomial interpolation, design a "parabolic method" which fits a quadratic polynomial through three function values x_{n-2}, x_{n-1} and x_n.
   Give an explicit formula for x_{n+1} in terms of $(x_n, f(x_n))$, $(x_{n-1}, f(x_{n-1})$, $(x_{n-2}, f(x_{n-2}))$.
2. (Computer) Implement your parabolic method in Mathematica (or other computer system) and use it to solve for π as the root of $sin(x)$. Compare the number of correct digits in your method to the number of correct digits in the same iteration of the secant method. Which is better? Download the worksheet "Comparison of Newton's Method and the Secant Method" to get started.
3. Newton's method assumes that the first derivative of the function is available. Suppose that the first *and second* derivatives of your function are available. Design a second order Newton's method which uses $f''(x_n)$ as well as $f(x_n)$ and $f'(x_n)$ to predict the location of the root of f. Compare your method to Newton's method in a specific case and show that it converges faster.
4. (Challenge) Follow the model of Theorem 1 on page 94 of the book to find and prove the order of convergence of the second order Newton's method. (It should be faster than the original Newton's method, of course.)
5. On page 173, we give an approximation to the second derivative of a function by

$$f''(x) \sim \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

and we show that the order of converge is $O(h^2)$. Use one step of Richardson extrapolation to improve this formula for $f''(x)$.

6. (Computer) Implement your formula in Mathematica and show that it converges faster than the original formula and gives more digits of accuracy before it succumbs to roundoff error. Use the Mathematica notebook "Richardson extrapolation and derivatives" as a model.

7. On page 173, we get an $O(h^4)$ formula for the first derivative with leading error term
$$-\frac{1}{30}f^{(5)}(\xi)h^4.$$
by fitting a third degree polynomial to $f(x)$ at the points $\{x-2h, x-h, x+h, x+2h\}$. In practice, we saw this was very comparable to the $O(h^4)$ estimate obtained by Richardson extrapolation (on page 167). We can improve this estimate by choosing the four points at which to interpolate f to be Chebyshev nodes on the interval $[x-2h, x+2h]$.
   - Derive the $O(h^4)$ approximation formula for $f'(x)$ using the Chebyshev nodes.
   - Show that in practice this estimate is slightly more accurate than the $O(h^4)$ formula given on page 173.