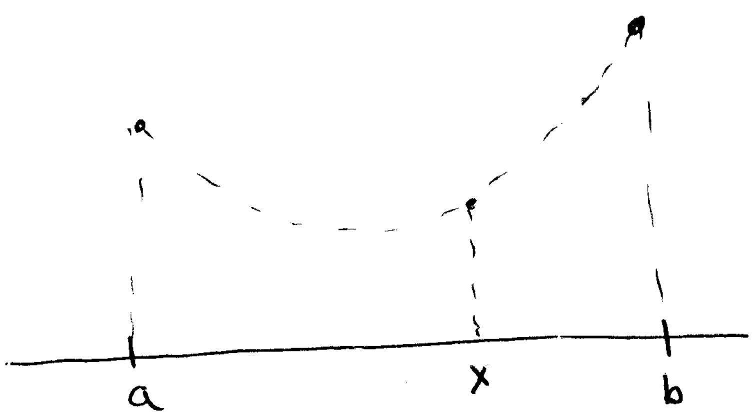
Brent's Method

This material is from Brent's really beautiful treatment in "Algorithms for Minimization without derivatives". (on webpage, since the book is hard to find).

Lemma. Suppose $x \in (a,b)$ and $f(a) > f(x)$, $f(b) > f(x)$. Then (a,b) contains a local minimum for f.



Idea: We want to alternate between quadratic interpolation steps and Golden section search steps so as to guarantee convergence always, but step faster if possible.

Definition. We keep track of six points

$[a,b]$:= an interval containing the local min

x := the point where the lowest value of f (so far) has been found

w := the point with the next lowest value of f (so far)

v := the last value of w

u := the last point at which f was evaluated

2a

The user is required to specify some tolerance tol, which is usually around $\sqrt{\text{machine eps}}$ (and should never be smaller).

(left blank on purpose)

Initial step:

u is undefined.

$$v = w = x = a + \left(\frac{3-\sqrt{5}}{2}\right)(b-a)$$

Generic step.

Let $m := \frac{1}{2}(a+b)$ be the midpoint of (a,b).

If x is within tol of a or b, quit.

Compute p, q so that

$x + p/q$ = minimum of parabola through $(x, f(x)), (v, f(v)), (w, f(w))$.

Let e = value of p/q at second-to-last step.

We reject quadratic interpolation if

$|e| < tol$

$x + (p/q) \notin (a,b)$

$|p/q| \geq \frac{1}{2} e.$

If quadratic interpolation is rejected

$$u = \begin{cases} \left(\frac{\sqrt{5}-1}{2}\right)x + \left(\frac{3-\sqrt{5}}{2}\right)a & \text{if } x \geq m \\ \left(\frac{\sqrt{5}-1}{2}\right)x + \left(\frac{3-\sqrt{5}}{2}\right)b & \text{if } x < m \end{cases}$$

Otherwise, ~~if 2|u-x|, u-a, b-u~~

$u = x + p/q,$ ~~as long as~~

as long as $|u-x|, u-a, b-u$ are all at least tol.

If one of these conditions is violated, we change u to enforce it.

Then we update a, b, x, w, v.

If $f(w) \leq f(x)$

x will be an endpoint of the new interval, which is $[a, x]$ or $[x, b]$ (depending on which one contains w).

In the next round,

$x = u_{old}, w = x_{old}, v = w_{old}$

If $f(u) > f(x)$

u will be an endpoint of the new interval, which is $[a,u]$ or $[u,b]$ depending on which one contains x

If $f(u) \leq f(w)$

we move $w = u_{old}$, $v = w_{old}$ for the next step

If not, and $f(u) \leq f(v)$

we move $v = u_{old}$ for the next step (and leave, x, w alone).

Lemma. Brent's method always converges.

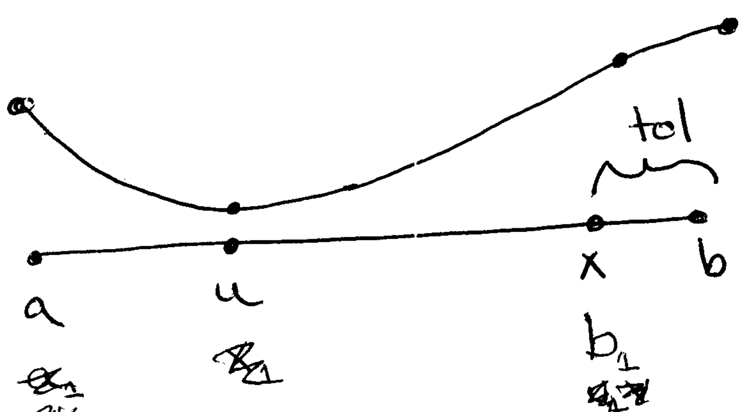
Observe that $[a,b]$ always contains x so that $f(a), f(b) > f(x)$. So we can't lose the minimum. The question is how fast the interval shrinks.

Now the correction $|P/q|$ must shrink by at least a factor of 2 at each iteration to avoid triggering a Golden section step.

On the other hand, once $|P/q| < \text{tol}$, we do a Golden section step anyway.

Therefore, we do a Golden section step every $2\log_2[(b-a)/tol]$ steps.

Now the worst case for Golden section steps is



where $f(u) < f(x)$. In this case $b \to x$, and the interval shrinks by only tol. But then $x \to u$. ~~and the next step~~

Now on the next step, we are
fitting a parabola through $(u, f(u)) = (x, f(x))$
so the new parabolic min $x + p/q$
~~is < less than f(u)~~
has $f(x + p/q) < f(u)$ or $(|u - x| = tol)$.
if we accept the parabolic step.
In either case, u or a point w/in
tol of it becomes an endpoint.