Math 4250 - Review

We are going to start gently by recalling some facts about linear algebra and calculus.

Definitions.

A vector $\vec{v} \in \mathbb{R}^n$ is a list $\vec{v} = (v_1, \ldots, v_n)$.

$\vec{v} + \vec{w} = (v_1 + w_1, \ldots, v_n + w_n)$

$K\vec{v} = (Kv_1, \ldots, Kv_n)$

$\langle \vec{v}, \vec{w} \rangle$ or $\vec{v} \cdot \vec{w} = \sum v_i w_i$

The length of $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

A set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ forms a <u>basis</u> for $\mathbb{R}^n$ <=> every $\vec{w} \in \mathbb{R}^n$ can be written <u>uniquely</u> as $\vec{w} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ for some $c_1, \ldots, c_n$.

If we write

$$A = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$$

then $c_1\vec{v}_1 + \ldots + c_n\vec{v}_n = \vec{\omega} \iff \vec{c}$ is the solution to the matrix equation

$$A\vec{c} = \vec{\omega}$$

since

$$\begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 v_{11} + c_2 v_{21} + \ldots + c_n v_{n1} \\ \\ c_1 v_{1n} + \ldots \ldots + c_n v_{nn} \end{bmatrix}$$

$$= c_1\vec{v}_1 + \ldots + c_n\vec{v}_n.$$

Prop. $\vec{v}_1, \ldots, \vec{v}_n$ is a basis $\iff A$ is an invertible matrix $\iff \det A \neq 0$.

Definition. An $n \times n$ matrix is <u>orthogonal</u>
$\iff AA^T = I = A^T A$ (or $A^T = A^{-1}$).

Lemma. An $n \times n$ matrix A is orthogonal iff the column vectors $\vec{v}_1, \ldots, \vec{v}_n$ are unit length and pairwise orthogonal.

Proof.

$$A^T A = \begin{bmatrix} \longleftarrow \vec{v}_1 \longrightarrow \\ \\ \longleftarrow \vec{v}_n \longrightarrow \end{bmatrix} \begin{bmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} & \langle \vec{v}_i, \vec{v}_j \rangle & \\ & \underset{\text{in position } i,j}{\nearrow} & \end{bmatrix}$$

This last matrix of dot products (called the <u>Gramian</u>) is the identity iff the $\vec{v}_i$ are unit length (so $\langle \vec{v}_i, \vec{v}_i \rangle = 1$) and pairwise orthogonal (so $\langle \vec{v}_i, \vec{v}_j \rangle = 0$). $\square$

Note that if A is an orthogonal matrix and we want to write a vector $\vec{\omega}$ as a linear combination $c_1\vec{v}_1 + \cdots + c_n\vec{v}$ of the columns of A, it's easy!

$$A\vec{c} = \vec{\omega}$$

so

$$\vec{c} = A^{-1}\vec{\omega}$$

so

$$\vec{c} = A^T\vec{\omega} = \begin{bmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \\ \leftarrow \vec{v}_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow \\ \vec{\omega} \\ \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \langle \vec{v}_1, \vec{\omega} \rangle \\ \vdots \\ \langle \vec{v}_n, \vec{\omega} \rangle \end{bmatrix}.$$

Explicitly, if $\vec{v}_1, \dots, \vec{v}_n$ are orthonormal,

$$\vec{\omega} = \langle \vec{\omega}, \vec{v}_1 \rangle \vec{v}_1 + \dots + \langle \vec{\omega}, \vec{v}_n \rangle \vec{v}_n .$$

Example. Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is orthogonal.

We know

$$a_{11}^2 + a_{21}^2 = 1 \qquad a_{12}^2 + a_{22}^2 = 1 .$$

$$a_{11} a_{12} + a_{21} a_{22} = 0 .$$

Solving the last equation,

$$\frac{a_{11}}{a_{21}} = -\frac{a_{22}}{a_{12}}$$

which means that there is some λ so that

$$A = \begin{bmatrix} a_{11} & -\lambda a_{21} \\ a_{21} & \lambda a_{11} \end{bmatrix}$$

Further, this means

$$a_{11}^2 + a_{21}^2 = 1$$

$$\lambda^2 a_{21}^2 + \lambda^2 a_{11}^2 = 1$$

so

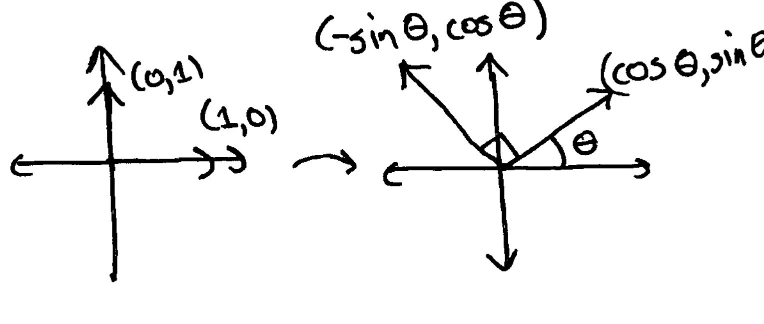
$$\lambda^2 = 1 \quad \text{and} \quad \underline{\lambda = \pm 1}.$$

This means there are two cases:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

Since $a^2 + b^2 = 1$, there is some angle θ so that $a = \cos\theta$, $b = \sin\theta$. Then

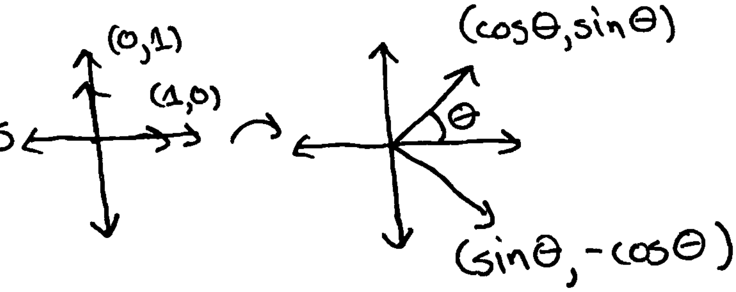
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ maps



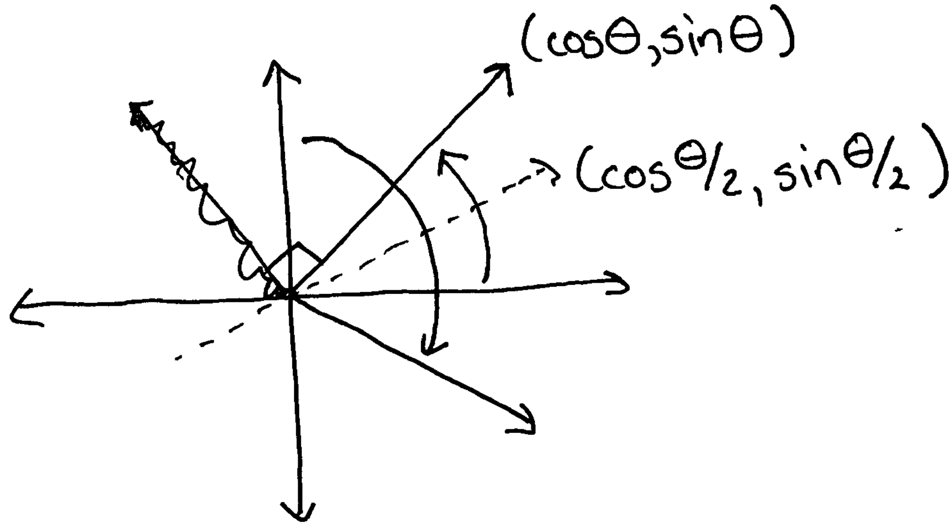
and we see this is a rotation by angle θ.

The other matrix,

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$ maps



turns out to be a reflection over the line $(\cos\theta/2, \sin\theta/2)$.



We won't prove this in class, as it's a homework exercise, but the picture should convince you a bit.

Definition. The <u>orthogonal</u> <u>group</u> $O(n)$ consists of all $n \times n$ orthogonal matrices. The <u>special</u> <u>orthogonal</u> <u>group</u> $SO(n)$ is the subgroup of $O(n)$ of matrices of determinant $+1$.

We can show

Proposition. Every matrix $A \in SO(3)$ is a rotation around some axis $\vec{v}$

(Proof. Homework, but the hint is that $\vec{v}$ is the eigenvector of A with eigenvalue 1.)

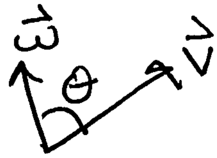
If you remember the cross product for vectors in $\mathbb{R}^3$,

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1)$$

It has the properties:

$$\langle \vec{v}, \vec{v} \times \vec{w} \rangle = \langle \vec{w}, \vec{v} \times \vec{w} \rangle = 0$$

$$\| \vec{v} \times \vec{w} \| = \| \vec{v} \| \, \| \vec{w} \| \sin \theta$$



$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

Proposition. The columns of a matrix A in SO(3) are in the form

$$A = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_1 \times \vec{v}_2 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Proof. (Again, homework!)

We close by recalling some useful identities:

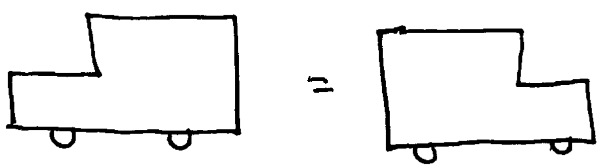
$$\langle \vec{a}, \vec{b} \times \vec{c} \rangle = \det \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a} & \vec{b} & \vec{c} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

is called the "triple product".

$$\langle \vec{a}, \vec{b} \times \vec{c} \rangle = \langle \vec{b}, \vec{c} \times \vec{a} \rangle = \langle \vec{c}, \vec{a} \times \vec{b} \rangle$$

(cyclic permutation)

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \langle \vec{a}, \vec{c} \rangle - \vec{c} \langle \vec{a}, \vec{b} \rangle$$



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