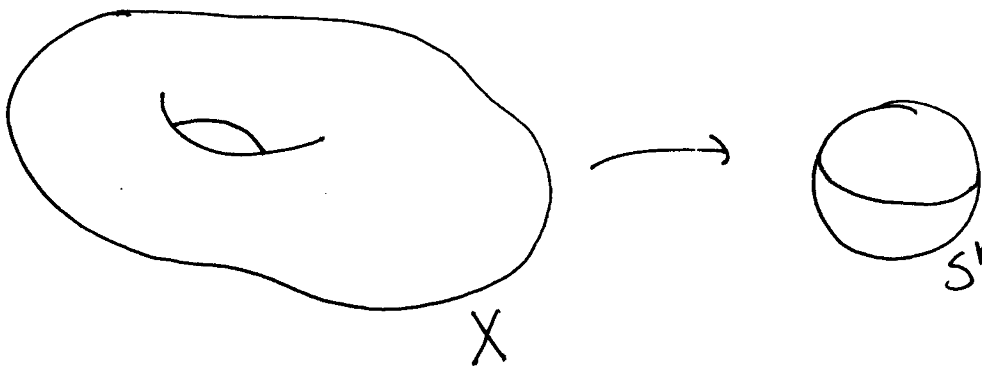
Concluding Lecture

The Gauss-Bonnet theorem and beyond.

Suppose now we have a hypersurface X (($K=2$) dimensional manifold) in $\mathbb{R}^{K+1}$.



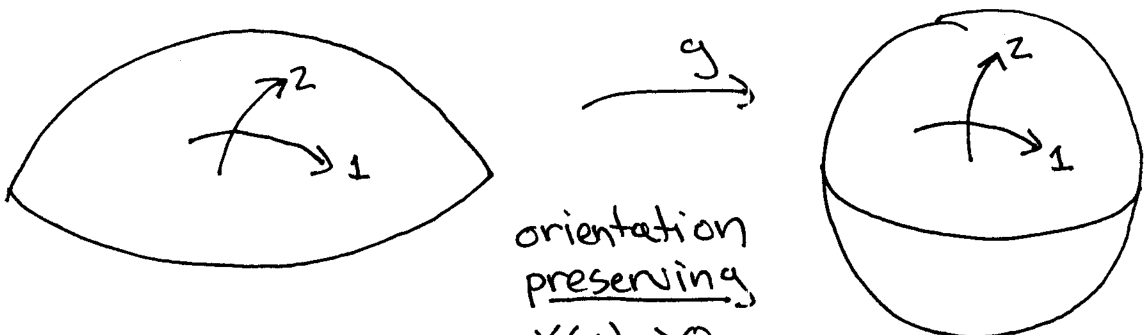
We can define a Gauss map $g: X \to S^K$ by orienting X and letting $g(x) = \vec{n}_x$, the unit (outward) normal to X at x.

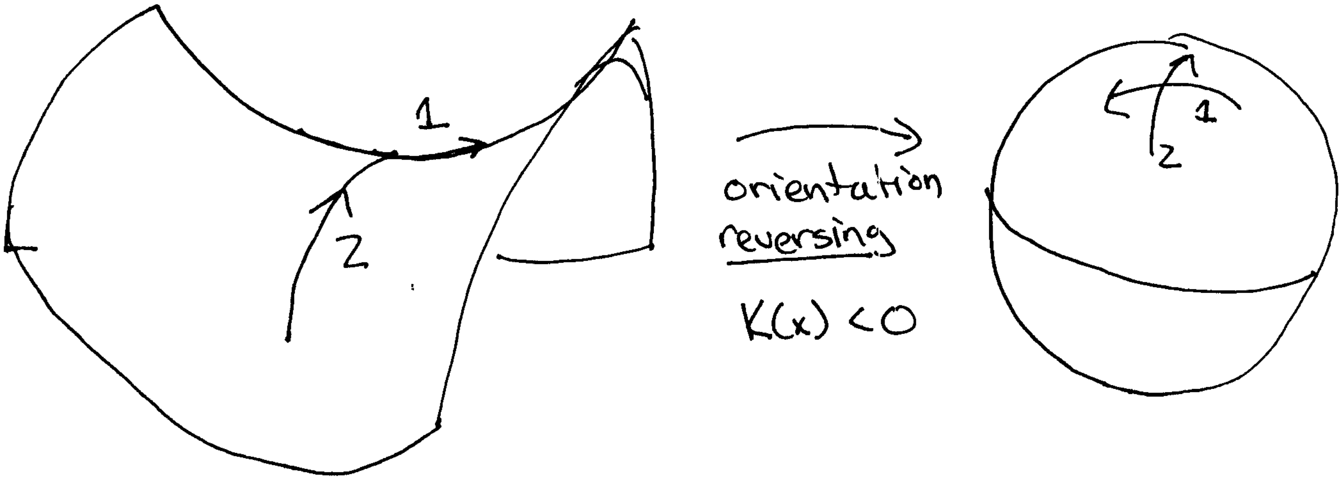
Definition. On any K-manifold in $\mathbb{R}^n$, we can define $V_X(x)$ on any $T_x X$ to be the alternating K-tensor with value $1/K!$ on each (positive) orthonormal basis for $T_x X$. Then the form V_X is called the <u>volume form</u> on X. It is also written $d\mathrm{Vol}_X$.

On our hypersurface X, we define

$$g^*(dVol_{S_k}) = \text{a } k\text{-form on } X$$
$$= f(x)\ dVol_X$$

and we call the function $f(x) = K(x)$ the __Gauss__ __curvature__ of X at x.





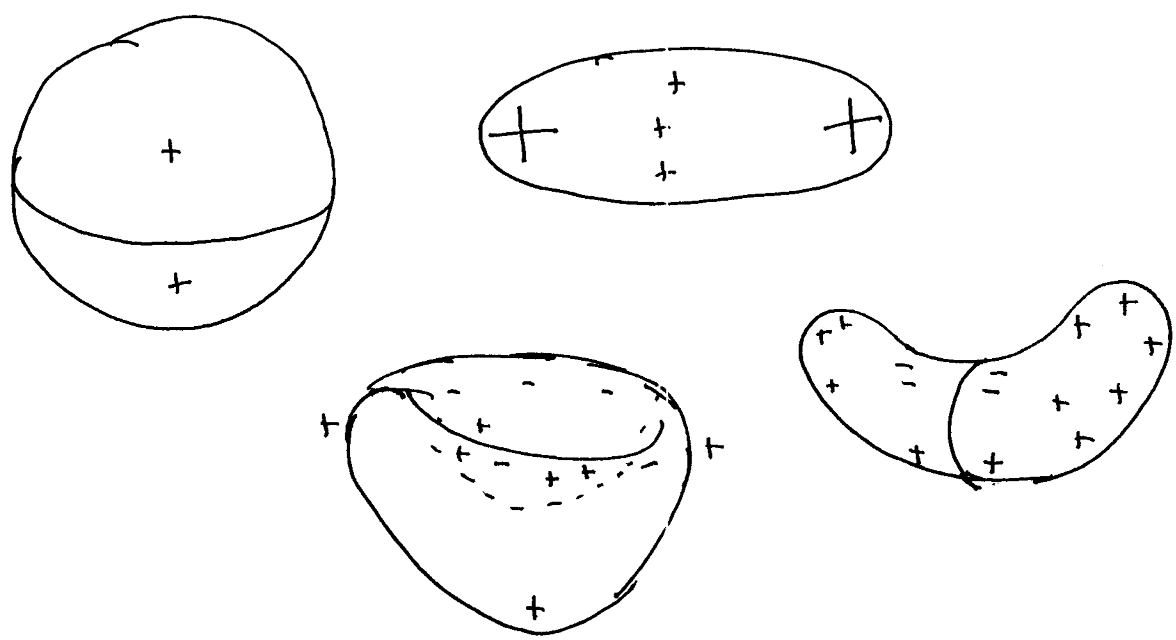
Now it is clear that

$$\int_X K(x)\, dVol_X = \int_X g^* dVol_{S^K}$$

$$= (\deg g) \int_{S^K} dVol_{S^K}$$

$$= (\deg g) \cdot vol(S^K).$$

and that this integral is a homotopy invariant of the surface! So all these integrals



are exactly the same!

<u>The Gauss-Bonnet Theorem.</u> If X is a compact even-dimensional hypersurface in $\mathbb{R}^{k+1}$ then

$$\int_X K(x)\, d\mathrm{Vol}_x = \frac{1}{2}\,\mathrm{vol}(S^k)\,\chi(X)$$

where $\chi(X)$ is the Euler characteristic.

Note: when X is odd-dimensional, this $=$ must be false (though $\int K(x)\,d\mathrm{Vol}_x$ is still an invariant) since $\chi(X)$ is zero.

We must show $\deg g = \frac{1}{2}\chi(X)$.

Idea. Choose $\vec{a} \in S^k$ so that $\pm a$ are regular values of g. Define a vector field on X by projecting $-\vec{a}$ to $T_x X$.

The zeros of this field are clearly the preimages of $\pm\vec{a}$. Let's work out an explict expression for our vector field.

$$\vec{V}(x) = (-\vec{a}) - [-\vec{a}\cdot\vec{n}(x)]\,\vec{n}(x)$$

$$= [\vec{a}\cdot g(x)]\,g(x) - \vec{a}.$$

If $T: \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is given by $T(x) = x - a$, then $\vec{V}(x) = T\circ[a\cdot g]g$.

Lemma. If $g(z) = a$, then $dv = dT\circ dg$ and if $g(z) = -a$, then $dv = -dT\circ dg$.

Proof. We must compute for any $\omega_z \in T_z X$

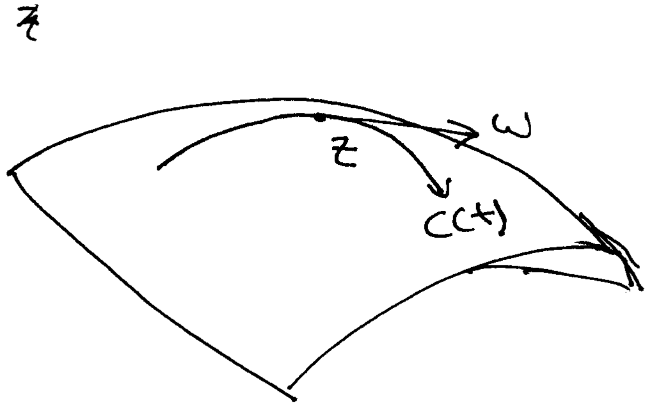
$$\cancel{d\big([a\cdot g(z)]\,dg(z)\big) = [a\cdot g(z)]\,d}$$

$$\cancel{d\big([a\cdot g(z)]\,g(z)\big)(\omega) =}$$

$$\cancel{d\big([a\cdot g(z)]\,g(z)\big) = d\big([a\cdot g(z)]g_1(z), \dots, [a\cdot g(z)]g_{k+1}(z)\big)}.$$

Suppose we let $f(x) = [a \cdot g(x)]\, g(x)$, and we compute

$$df_z(\omega).$$



By previous definition, if $c(t)$ is a curve in X with $c'(0) = \omega$, $c(0) = z$ we have

$$df_z(\omega) = \frac{d}{dt} f(c(t))\Big|_{t=0}$$

$$= \frac{d}{dt} [a \cdot g(c(t))]\, g(c(t)) \Big|_{t=0}$$

$$= [a \cdot g'(c(t))]\, g(c(t)) + [a \cdot g(c(t))]\, g(c(t))' \Big|_{t=0}$$

Now $g(c(0)) = g(z) = \pm a$, so the second term is $\pm \frac{d}{dt} g(c(t))\Big|_{t=0} = \pm dg_z(\vec{\omega})$. The first term involves

$$a \cdot g'(c(t)), \text{ but } g'(c(t)) \in T_a S^k \perp \vec{a}$$

so it vanishes.

Corollary. The index of $\vec{v}$ at z is $+1$ if $g: X \to S^k$ preserves orientation and -1 if $g: X \to S^k$ reverses orientation.

Proof. It turns out to be the case that at a zero, $d\vec{v}: T_z X \to T_z X$ and the index of the zero is ± 1 depending on whether $d\vec{v}$ preserves/reverses orientation.

If $g(z) = a$, then

$$d\vec{v} = dT \circ dg.$$

We know dT is orientation preserving, and the lemma follows.

If $g(z) = -a$, then

$$d\vec{v} = dT \circ (-dg)$$

Now $\det(-dg) = (-1)^k \det dg$, but k is even so the theorem follows.

Now adding up indices, we get

$$\chi(X) = \underset{\substack{\uparrow \\ \text{sum at } a}}{\deg g} + \underset{\substack{\uparrow \\ \text{sum at } -a}}{\deg g}$$

$$= 2 \deg g$$

as desired!

Comment. In fact, we can define curvature "intrinsically" for k-manifolds which aren't hypersurfaces. The G-B theorem holds here too, but the proof is a lot harder.

We have an impressive toolkit: what next?

1) ~~For~~ For any manifold X, the space of k-forms contains two subspaces:

<u>closed</u> forms have $d\omega = 0$

<u>exact</u> forms are $d\alpha = \omega$

We see that

$$\text{closed}/\text{exact} \text{ is well-defined.}$$

In fact, it is a finite-dimensional ~~sub~~space which tells you about the topology of X: $H^k(X, \mathbb{R})$.

2) Characteristic classes.

These spaces ~~cont~~ $H^0(X,\mathbb{R}), \dots, H^n(X,\mathbb{R})$ contain certain classes which tell us more about X. We have seen one, the Euler class χ which gives us

$$\int_X \chi = \chi(X)$$

There are others...

3) We can describe the space of k-forms better with the Hodge star

$$* : \Lambda^k(X) \to \Lambda^{n-k}(X)$$

defined by

$$dx_I \wedge * dx_I = dx_1 \wedge \dots \wedge dx_n .$$

Then if

$$\delta : \Lambda^k(X) \to \Lambda^{k+1}(X)$$

is given by $*d*$, we have

$$\Lambda^k(X) = d\alpha \oplus \delta\beta \oplus H^k(X)$$

and this is orthogonal with respect to the Hodge inner product on Λ^k,

$$\langle \alpha, \beta \rangle = \int \alpha \wedge (*\beta).$$

I could go on... but that's a topic for my next class (8120, S'09). See yoo there!