Stokes' Theorem

We now have two classes of objects: manifolds and forms, related by the operation of integration.

$$
\begin{tikzcd}
\text{oriented } K\text{-manifolds} \arrow[d, "\partial"] & \int \arrow[l] \arrow[r] & K\text{-forms} \\
K\text{-1 manifolds with } \partial \text{ orientation} & \int \arrow[l] \arrow[r] & K\text{-1 forms} \arrow[u, "d"']
\end{tikzcd}
$$

It is amazing that our constructions lock together to yield:

Stokes Theorem. If ω is any smooth K-1 form on the K-manifold X, then

$$\int_{\partial X} \omega = \int_X d\omega.$$

Proof. Both sides are linear in ω, so we can take a partition of unity and assume that ω is supported on a coordinate patch $h: U \to X$.

If $U \subset \mathbb{R}^k$ is open (that is, our patch avoids the boundary). Now we have

$$\int_{\partial X} \omega = 0 \qquad \text{(since } \omega \text{ is supported on } U\text{)}$$

and

$$\int_X d\omega = \int_U h^*(d\omega) = \int_U d(h^*\omega)$$

Let $\nu = h^*\omega$. This is a $(k-1)$-form on $\mathbb{R}^k$, so

$$\nu = \sum (-1)^{i-1} f_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k$$

Now then

$$d\nu = \sum_i \frac{\partial f_i}{\partial x_i} \, dx_1 \wedge \ldots \wedge dx_k$$

So

$$\int_{\mathbb{R}^k} dv = \sum_i \int_{\mathbb{R}^k} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_k$$

By Fubini's theorem, the rhs is an iterated integral which can be taken in any order - let's start with

$$\int_{\mathbb{R}^k} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_k = \int_{\mathbb{R}^{k-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_k$$

but we know $f = 0$ outside U, so the inner integral is 0, as required.

What if U intersects ∂X? We can follow above argument except for the term involving $\frac{\partial f_k}{\partial x_k}$. We have

$$\int_{\mathbb{H}^k} dv = \int_{\mathbb{R}^{k-1}} \left(\int_0^{\infty} \frac{\partial f_k}{\partial x_k} dx_k \right) dx_1 \cdots dx_{k-1}$$

As above, $f_k(x_1,\dots,x_k) = 0$ for large k, but we know

$$f_k(x_1,\dots,x_{k-1},0) \neq 0.$$

So

$$\int_{\mathbb{H}^k} d\nu = \int_{\mathbb{R}^{k-1}} -f_k(x_1,\dots,x_{k-1},0)\, dx_1 \cdots dx_{k-1}$$

Now

$$\int_{\partial X} \omega = \int_{\partial H^k} h^* \omega = \int_{\partial H^k} \nu$$

So what is this form? Well, under inclusion,

$$i^* dx_i = dx_i \quad \text{for } i \in 1,\dots,k-1$$
$$i\ dx_k = 0$$

so the $k-1$ form ν on ∂H_k is

$$\nu = (-1)^{k-1} f(x_1,\dots,x_{k-1},0)\, dx_1 \wedge \dots \wedge dx_{k-1}$$

and the integral

$$\int_{\partial H^k} \omega = \int (-1)^{k-1}$$

so

$$(\psi^{-1})^* d(\psi^*\omega) = (\psi^{-1})^* g^* d(\varphi^*\omega)$$
$$= (g \circ \psi^{-1})^* d(\varphi^*\omega)$$
$$= (\varphi^{-1})^* d(\varphi^*\omega).$$

so (as expected) this doesn't depend on the choice of coordinates.

All of the previous properties hold for d on manifolds. So we can show

Theorem. Let $g: Y \to X$ be any smooth map of manifolds (which may have boundary). Then for any ω on X,

$$d(g^*\omega) = g^*(d\omega).$$