Integration on manifolds.

We recall that we "know" how to change variables when integrating in multivariable calculus.

Theorem. If $f: V \to U$ is a diffeomorphism of open sets in $\mathbb{R}^k$ and a is an integrable function on U. Then

$$\int_U a \; dx_1 \cdots dx_k = \int_V (a \circ f) \, |\det df| \, dy_1 \cdots dy_k.$$

Now we claimed a few minutes ago that if

$$\omega = a \, dx_1 \wedge \cdots \wedge dx_k$$

then

$$f^* \omega = (a \circ f) \det df \; dy_1 \wedge \cdots \wedge dy_k.$$

So this theorem <u>really</u> says

$$\int_U \omega = \int_V f^* \omega.$$

Now, that was amazing - the missing determinant came from the pullback process itself, or, if you like, from the anticommutativity property of alternating tensors.

We now note that

a) we proved a lot of stuff about orientation by appealing to properties of det

b) ~~det~~ these properties are now revealed to be properties of alternating tensors.

So it isn't surprising that orientation, integration and forms all come together in:

Theorem. Assume that $f: U \to V$ is a diffeo. of open sets in $\mathbb{R}^k$ or H^k and let ω be an integrable k-form on U. Then

$$\int_U \omega = \pm \int_V f^* \omega$$

and the sign depends on whether f preserves or reverses orientation.

Example. Suppose $g : \mathbb{R} \to \mathbb{R}$ is an integrable function, $U = [a,b]$ and $V = [a,b]$ with

$f(y) = -y + (b+a)$.

We see $f(a) = b$ and $f(b) = a$, while

$$df = [f'(x)] = [-1]$$

has determinant -1. So this is orientation <u>reversing</u>

$$\int_U g\,dx = -\int_V f^*(g\,dx) = -\int_V g(f(y))\cdot(-1)\,dy$$

Now

$$\int_U g\,dx = \int_a^b g(x)\,dx$$

But

$$\int_V g(f(y))\,dy = \int_a^b g(f(y))\,dy = -\int_b^a g(f)\,df$$

since

$$f = f(y), \text{ so } df = f'(y)\,dy = -dy.$$

so

$$\boxed{\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx}$$

$f(a) = b$ $\quad$ (a,b)

$f(b) = a$ $\quad$ (b,a)

$m = \dfrac{a-b}{b-a} = -1$

$f(a) = ma + c$

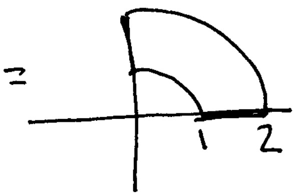
$b = -1a + c$

$b + a = c$

$f(x) = -x + (b+a)$.

That was rather abstract, so let's try

Example. $U =$ $\qquad\qquad V = (1,2)\times(0,\pi/2)$

$=$ 

$$f(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$\omega = dx \wedge dy.$$

We compute

$$df = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}, \quad \det df = r > 0$$

Since r is orientation preserving

$$\int_U dx \wedge dy = \int_V f^*(dx \wedge dy)$$

$$= \int_V r \, dr \wedge d\theta$$

by the determinant theorem.

But if we pull back using our previous formulae directly?

$$f^* dx = \sum_{j=1}^{2} \frac{\partial f_1}{\partial y_j}\, dy_j$$

$$= \frac{\partial f_1}{\partial r}\, dr + \frac{\partial f_1}{\partial \theta}\, d\theta$$

$$= \cos\theta\, dr - r\sin\theta\, d\theta$$

$$f^* dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

So

$$f^*(dx \wedge dy) = f^* dx \wedge f^* dy$$

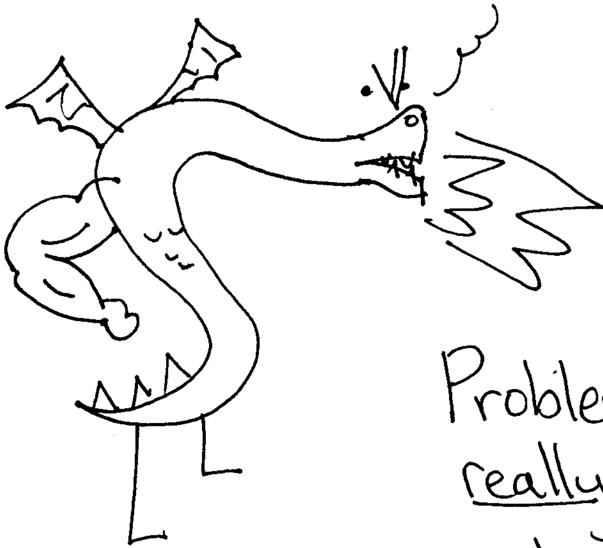
$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr$$

$$= r(\cos^2\theta + \sin^2\theta)\, dr \wedge d\theta.$$

$$= r\, dr \wedge d\theta.$$

which just has to make you feel better about this theory...



Problem. If integration is <u>really</u> Riemann summation, and Riemann sums depend on coordinates, how do we define a coordinate-free version of integration? on manifolds?

Well, we wouldn't even pose this question if the answer wasn't forms. Here's how.

1.) Given a form ω, write it as $\omega = \sum p_i \omega$ where the p_i are a partition of unity so each p_i is supported in a parametrizable open subset of X.

2) For each $p_i\omega$, parametrize $W \subset X$ supporting $p_i\omega$ by $U \subset \mathbb{R}^k$ via

$$h : U \to W$$

<u>Define</u>

$$\int_X p_i\omega = \int_U h^*(p_i\omega)$$

Is this independent of h? Well, take

$$\begin{tikzcd} & W & \\ U \arrow[ur, "h"] & & V \arrow[ul, "g"'] \arrow[ll, "f = h^{-1}\circ g"] \end{tikzcd}$$

Observe

$$\int_U h^*\omega = \int_V f^* h^* \omega$$
$$= \int_V (h \circ f)^* \omega$$
$$= \int_V g^* \omega.$$

3) Now <u>define</u> $\int \omega = \sum_i \int p_i \omega$.

Does this depend on p? Well, suppose p_j' was another partition: For each $p_i\omega$,

$$\int_X p_i \omega = \sum_j \int_X p_j' p_i \omega$$

since $\sum_j p_j' = 1$ everywhere, and integration is linear (linearity of integration in $\mathbb{R}^k$ and of pullback). But then also

$$\int_X p_j' \omega = \sum_i \int_X p_i p_j' \omega$$

so

$$\begin{aligned} \sum_i \int_X p_i \omega &= \sum_i \sum_j \int_X p_j' p_i \omega \\ &= \sum_j \sum_i \int_X p_i p_j' \omega \\ &= \sum_j \int_X p_j' \omega, \end{aligned}$$

as desired.

We finally have

Theorem. If $f: Y \to X$ is an orientation preserving diffeomorphism then

$$\int_X \omega = \int_Y f^*\omega$$

for all (compactly supported) smooth K-forms on X where $K = \dim X = \dim Y$.

Next time: Integrating forms with dimension $< K$.

Example

Consider the 1-form

$$\omega = \sin x \, dx + \cos y \, dy$$

and the map $\mathbb{R}^3 \to \mathbb{R}^2$ given by $g(x,y,z) = (xy, yz)$ and compute $g^*\omega$.