Differential forms (IV)

We have now defined a space of alternating p-tensors $\Lambda^p(V^*)$ for any vector space V.

We can now define differential forms!

Definition. Let X be a smooth manifold. A <u>p-form</u> on X is a function ω which assigns to each $x \in X$ an alternating p-tensor $\omega(x)$ on $T_x X$.

Some properties of forms are immediate:

1. forms can be added and scalar-multiplied pointwise
2. forms can be wedged pointwise, and if ω is a p-form, θ is a q-form

$$\omega \wedge \theta = (-1)^{pq}\, \theta \wedge \omega$$

Examples.

Any function $\varphi : X \to \mathbb{R}$ is a 0-form.

Given a smooth $\varphi : X \to \mathbb{R}$, the differential $d\varphi_x : T_x X \to \mathbb{R}$ is a linear functional (or alternating 1-tensor) on each $T_x X$. Thus the differential defines a 1-form $d\varphi$.

The coordinate functions

$$\begin{matrix} x_1 : \mathbb{R}^k \to \mathbb{R} & & dx_1 \\ \vdots & \text{define 1-forms} & \vdots \\ x_k : \mathbb{R}^k \to \mathbb{R} & & dx_k \end{matrix}$$

We see that at any point $z \in \mathbb{R}^k$,

$$\left(dx_j(z)\right)\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = v_j$$

So these 1-forms define a standard basis for p-forms on $\mathbb{R}^n$: if $I = i_1, \ldots, i_p$ then

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

and the basis consists of all dx_I with increasing sequences I.

Proposition. Every p-form on $U \subset \mathbb{R}^k$ can be written uniquely as

$$\sum_I f_I \, dx_I$$

where the f_I are functions on $\cancel{\mathbb{R}}$ U.

We now do a sanity check:

Lemma. Given a smooth $\varphi : U \to \mathbb{R}$,

$$d\varphi = \sum \frac{\partial \varphi}{\partial x_i} dx_i$$

Proof. Both sides are linear functionals on $\mathbb{R}^k$. So we check for each $\vec{v} \in \mathbb{R}^k$,

$$\begin{aligned} d\varphi(\vec{v}) &= \langle \nabla \varphi, \vec{v} \rangle \\ &= \sum \frac{\partial \varphi}{\partial x_i} v_i \\ &= \sum \frac{\partial \varphi}{\partial x_i} dx_i(\vec{v}). \end{aligned}$$

The transpose operation of last class induces a natural way to transfer forms under smooth maps.

Construction. If $f: X \to Y$ is a smooth map and ω is a p-form on Y, then we define

$$\underbrace{f^*\omega(x)}_{\text{an alternating } p\text{-tensor on } T_x X} = \underbrace{\overset{\text{transpose of linear map } df_x}{(df_x)^*}\overbrace{(\omega(f(x)))}^{\text{an alternating } p\text{-tensor on } T_{f(x)}Y}}$$

So the pullback form $f^*\omega$ on X is given by the tensor field above. In words, given $v_1, \dots, v_p \in T_x X$

$$f^*\omega(v_1, \dots, v_p) = \omega(df_x v_1, \dots, df_x v_p).$$

We claim

Proposition.

$$f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$$

$$f^*(\omega \wedge \theta) = f^*(\omega) \wedge f^*(\theta).$$

$$(f \circ h)^* \omega = h^* f^* \omega.$$

Proof. The previous explanation for f^* makes the first two easy. The third is really the statement that given $X \xrightarrow{f} Y \xrightarrow{h} Z$

$$(df_y \circ dh_{f(x)})^* = (dh_{f(x)})^* (df_x)^*$$

which is easy to check.

We now want to write f^* explicitly in coordinates.

Given $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$, let $f : V \to U$ be smooth. We let

$x_1, \dots, x_k$ be coordinates on U
$y_1, \dots, y_\ell$ be coordinates on V

If

$$f = (f_1(y_1, \dots, y_\ell), \dots, f_k(y_1, \dots, y_\ell))$$

then

$$df_y = \left(\frac{\partial f_i}{\partial y_j}(y) \right)$$

and

$$(df_y)^* = \left(\frac{\partial f_j}{\partial y_j}(y) \right)^T$$

← the transpose matrix

So

$$f^* dx_i = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & & \frac{\partial f_k}{\partial y_1} \\ & & \\ \frac{\partial f_1}{\partial y_\ell} & \cdots & \frac{\partial f_k}{\partial y_\ell} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \sum \frac{\partial f_i}{\partial y_j} dy_j$$

Now we call this

$$df_i = \sum_{j=1}^{\ell} \frac{\partial f_i}{\partial y_j} dy_j$$

So using linearity → distributivity over $\wedge$. we can write for

$$\omega = \sum_I a_I \, dx_I$$

that

$f^* a_I(x) = a_I(f(y))$.

$$f^* \omega = \sum_I (f^* a_I) \, df_I$$

where if $I = i_1, \ldots, i_p$, we have

$$df_I = df_{i_1} \wedge \ldots \wedge df_{i_p}.$$

This theorem <u>looks</u> great, but <u>is</u> not terribly useful. We give a special case.

Suppose $k = \ell$ and $\omega = dx_1 \wedge \ldots \wedge dx_k$ (the "volume form" on $\mathbb{R}^k$). Then

$$f^* \omega = (\det df_y) \, \omega$$