Differential forms (III)

We have now defined $\Lambda^p(V^*)$, the alternating p-forms, and the wedge product $\wedge : \Lambda^p(V^*) \times \Lambda^q(V^*) \to \Lambda^{p+q}(V^*)$.

We want to define a basis for $\Lambda^p(V^*)$ and compute its dimension.

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Our goal now is to derive a basis for $\Lambda^p(V^*)$, using the wedge product.

~~Claim~~. If T is any p-tensor, we recall that a basis $\varphi_1, \dots, \varphi_k$ for V^* yields a basis for $\mathcal{J}^p$ in the form

$$T = \sum t_{i_1, \dots, i_p} \phi_{i_1} \otimes \dots \otimes \phi_{i_p}$$

So if T altermates,

$$T = \mathrm{Alt}(T) = \sum t_{i_1, \dots, i_p} \mathrm{Alt}(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_p})$$

$$= \sum t_{i_1, \dots, i_p} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_p}.$$

So the set of p-tensors in the form $\varphi_{i_1} \wedge \dots \wedge \varphi_{i_p}$ span $\Lambda^p(V^*)$. We will see they are <u>not</u> independent.

We observe that on one-tensors φ, ψ,

$$\varphi \wedge \psi = \frac{1}{2}(\varphi \otimes \psi - \psi \otimes \varphi)$$

and so

$$\varphi \wedge \psi = -(\psi \wedge \varphi),$$

so

$$\boxed{\wedge \text{ is anticommutative on 1-forms.}}$$

This means that if $I = i_1, \dots, i_p$ and $J = j_1, \dots, j_p$ differ only in ordering, then

$$\varphi_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_p} = \pm \varphi_J = \varphi_{j_1} \wedge \dots \wedge \varphi_{j_p}.$$

Further, if $i_x = i_y$ for any pair of indices in I, then $\varphi_I = 0$.

Thus if we want to pick a set of I so that the φ_I are <u>independent</u>, we should assume

$$1 \leq i_1 < i_2 < \ldots < i_p \leq K$$

The number of such sequences is $\binom{K}{p} = \frac{K!}{p!\,(K-p)!}$.

Lemma. All such tensors are independent.

Proof. Let $\{v_1, \ldots, v_K\}$ be the basis for V dual to $\{\varphi_1, \ldots, \varphi_K\}$. For any increasing index sequence I, observe

$$\varphi_I(v_I) = \mathrm{Alt}(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_p})(v_I)$$

$$= \frac{1}{p!} \sum_{\pi \in S_p} (\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_p})^{\pi}(v_I)$$

Of these permutations, $\pi = e$ returns 1, but all others feed some v_x to a φ_y with a different subscript, so

$\varphi_I(v_I) = 1$. Similarly, $\varphi_J(v_I) = 0$ if $I \neq J$

and J is also increasing.

Now suppose

$$\sum a_I \Phi_I = 0$$

for a linear combination of φ_I. Observing that this implies for any incr. sequence J

$$\sum a_I \varphi_I(v_J) = \frac{1}{p!} a_J = 0,$$

we see each coefficient $a_I = 0$, proving the claim.

Corollary. If $\{\varphi_1, \ldots, \varphi_k\}$ is a basis for V^*, then $\{\varphi_I = \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_p},\ 1 \leq i_1 < i_2 < \ldots < i_p \leq k\}$ is a basis for $\Lambda^p(V^*)$. Thus

$$\dim \Lambda^p(V^*) = \binom{k}{p}.$$

Observe that if $\varphi_I \in \Lambda^p$ and $\varphi_J \in \Lambda^q$, then writing each as a $\wedge$ of elements in Λ^1, we see

$$\varphi_I \wedge \varphi_J = (-1)^{pq} \varphi_J \wedge \varphi_I .$$

Consequence: If $\varphi_I \in \Lambda^2$, then $\varphi_I^2 = \varphi_I \wedge \varphi_I$ may be meaningful.

Interesting corollary: $\Lambda^{\dim V}(V^*)$ is 1-dimensional, and contains only scalar multiples of the determinant function!

Porism. If $p > k$, then $\Lambda^p(V^*)$ is 0-dimensional. (Any index sequence I repeats an index.)

It sometimes helps to define

$$\begin{aligned} \Lambda^0(V^*) &= \text{constant functions on } V \\ &= \mathbb{R} \end{aligned}$$

We define (for $f \in \Lambda^0(V^*)$),

$$f \wedge \varphi = f\varphi \quad \text{(scalar multiplication)}$$

Definition. Given a finite dimensional V,

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^k(V^*)$$

is a noncommutative algebra called the <u>exterior algebra</u> of V^*.

We have one more basic construction. If $A: V \to W$ is a linear map, then $A^*: W^* \to V^*$ is the associated transpose map. We can define

$$A^*: \Lambda^p(W^*) \to \Lambda^p(V^*)$$

by

$$A^*T(v_1, \ldots, v_p) = T(Av_1, \ldots, Av_p).$$

Lemma. A^* is linear, $A^*(T\wedge S) = A^*T \wedge A^*S$.

So we see that

$$\boxed{A^*: \Lambda^*(W^*) \to \Lambda(V^*) \text{ is an algebra homomorphism}}$$

Theorem. If $A: V \to V$ is a linear map, then $A^*T = (\det A)\,T$ for every $T \in \Lambda^{\dim V}(V^*)$. In particular, given $\varphi_1, \dots, \varphi_k \in \Lambda^1(V^*)$ (where $k = \dim V$), then

$$A^*\varphi_1 \wedge \dots \wedge A^*\varphi_k = (\det A)\, \varphi_1 \wedge \dots \wedge \varphi_k$$

Proof. We know $\Lambda^k(V^*)$ is one-dimensional, so $A^*T = \lambda T$ for some λ. Further, $\det \in \Lambda^k((\mathbb{R}^k)^*)$. So if we let $B: V \to \mathbb{R}^k$ be any isomorphism,

$$T = B^*(\det) \in \Lambda^k(V^*).$$

So

$$A^*T = A^*B^*(\det) = \lambda B^*(\det)$$

and

$$(B^*)^{-1} A^* B^* (\det) = \lambda (B^{*-1})(B^*)(\det)$$
$$= \lambda (BB^{-1})^* (\det) = \lambda \det.$$

Now each of these sides lies in $\Lambda^k((\mathbb{R}^k)^*)$.
We observe for ~~any basis~~ $e_1, \dots, e_k$, we have

$$\det(BAB^{-1}e_1, \dots, BAB^{-1}e_k) = \lambda \det(e_1, \dots, e_k)$$

so

$$\lambda = \det(BAB^{-1}) = \det A, \text{ as claimed.}$$