Differential Forms.

Have you ever integrated

$$\int f(x)\,dx$$

and wondered what dx really <u>was</u>?

Or noticed the similarity between

FTC $$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Stokes $$\int_{\partial S} \vec{V} \cdot ds = \iint_S (\nabla \times V) \cdot n \, dA$$

Divergence $$\int_{\partial \Omega} V \cdot n \, dA = \iiint_\Omega (\nabla \cdot V) \, d\mathrm{Vol}$$

If so, you have wanted to understand <u>differential</u> <u>forms</u>.

"Definition": A differential k-form is something you can integrate over a k-dimensional manifold.

Examples.

ds = arclength = 1 form

$(V \cdot n)\, dA$ = flux of vector field = 2 form

$dVol$ = volume = 3 form

If you have ever changed coordinates, you understand that forms have different expressions in different coordinate systems.

Example.

$$\int_{D\subset\mathbb{R}^2} d\text{Area} = \int_{D\subset\mathbb{R}^2} 1\, dx\, dy = \int_{D\subset\mathbb{R}^2} r\, dr\, d\theta$$

Goal: Understand K-forms as a way to understand K-manifolds more deeply.

Definition. A p-tensor on V is a function $T : V^p \to \mathbb{R}$, which is linear in each variable.

Example.

$f(\vec{v}) = v_1$, or any other linear functional on $\mathbb{R}^n$

$(\vec{v}, \vec{w}) \mapsto \langle \vec{v}, \vec{w} \rangle$, a 2-tensor on $\mathbb{R}^n$

$\vec{v}_1, \ldots, \vec{v}_k \mapsto \det \begin{pmatrix} \uparrow & & \uparrow \\ v_1 & \cdots & v_k \\ \downarrow & & \downarrow \end{pmatrix}$ a k-tensor on $\mathbb{R}^k$

Definition. If T is a p-tensor, S is a q-tensor, we let the $p+q$-tensor

$$T \otimes S (v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) = T(v_1, \ldots, v_p) \cdot S(v_{p+1}, \ldots, v_{p+q}).$$

Lemma. $\otimes$ is associative, distributes over addition, not commutative.

Observe that

$$V^* = \text{dual to } V = \mathcal{J}^1(V^*)$$

$\uparrow$ 1-tensors on V

We prove.

Theorem. Let $\{\varphi_1, \dots, \varphi_k\}$ be a basis for V^*, then the p-tensors

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_p}, \quad 1 \le i_1, \dots, i_p \le k$$

form a basis for $\mathcal{J}^p(V^*)$. Thus $\dim \mathcal{J}^p(V^*) = (\dim V)^p = k^p$.

Define. Alternating tensors

Differential Forms (cont)

Definition. A tensor is <u>alternating</u> if it reverses sign whenever two variables are transposed.

We recall the useful <u>fact</u>:

Any permutation of n letters can be written as a product of transpositions. The decomposition is <u>not</u> unique, but all decompositions have even or odd # of transpositions.

Definition. Let $(-1)^\pi$ be $+1$ if π has an <u>even</u> # of transpositions, -1 if π has an odd number of transpositions.

Lemma. T is alternating $\Leftrightarrow$

$$T^\pi(v_1, \dots, v_n) := T(v_{\pi(1)}, \dots, v_{\pi(n)}) = (-1)^\pi T(v_1, \dots, v_n).$$

We can construct an alternating tensor from any tensor T by averaging:

$$\mathrm{Alt}(T) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi T^\pi$$

Lemma. $\mathrm{Alt}(T)$ is alternating.

Observe $(-1)^{\pi \circ \sigma} = (-1)^\pi (-1)^\sigma$, since $\pi \circ \sigma$ has (# of transpositions in π) + (# of transpositions in σ) transpositions.

So

$$[\mathrm{Alt}(T)]^\sigma = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi (T^\pi)^\sigma$$

$$= \overbrace{(-1)^\sigma \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi (-1)^\sigma}^{\text{product is } 1} T^{\pi \circ \sigma}$$

$$= (-1)^\sigma \frac{1}{p!} \sum_{\pi \in S_p} (-1)^{\pi \circ \sigma} T^{\pi \circ \sigma}$$

Note: If T is alternating $\mathrm{Alt}(T) = T$.

We now observe

Lemma. Alternating p-tensors form a vector subspace $\Lambda^p(V^*)$ of p-tensors $\mathcal{T}^p(V^*)$

Proof. Just check sums and scalar multiples.

There is no reason to believe that $T \otimes S$ is alternating if T, S alternate (What if the transposition swaps a vector in the first group for one in the second?)

Definition. If $T \in \Lambda^p(V^*)$, $S \in \Lambda^q(V^*)$, let

$$T \wedge S = \mathrm{Alt}(T \otimes S) \in \Lambda^{p+q}(V^*).$$

We call this the <u>wedge</u> product.

Lemma. Wedge distributes over addition and scalar multiplication.

Problem. Is it obvious that $\wedge$ is associative? No!

Lemma. If $\mathrm{Alt}(T) = 0$, then $T \wedge S = 0 = S \wedge T$.

Proof. Consider S_{p+q}, which a subgroup $G \cong S_p$ which permutes only the first p elements of $(1, \ldots, p+q)$. Let $\pi \in G \mapsto \pi' \in S_p$ under the isomorphism.

Note: $(T \otimes S)^{\pi} = T^{\pi'} \otimes S$, and $(-1)^{\pi} = (-1)^{\pi'}$.

Now

$$\sum_{\pi \in G} (-1)^{\pi} (T \otimes S)^{\pi} = \Big[\sum_{\pi' \in S_p} (-1)^{\pi'} T^{\pi'} \Big] \otimes S$$

$$= \mathrm{Alt}(T) \otimes S = 0.$$

Further, G decomposes S_{p+q} into cosets $G \circ \sigma = \{ \pi \circ \sigma : \pi \in G \}$.

For each coset

$$\sum_{\pi\circ\sigma \in G\circ\sigma} (-1)^{\pi\circ\sigma} (T\otimes S)^{\pi\circ\sigma} = (-1)^{\sigma}\Big[\underbrace{\sum_{\pi\in G} (-1)^{\pi} (T\otimes S)^{\pi}}_{0}\Big]^{\sigma}$$

$$= 0.$$

But $Alt(T\otimes S)$ is the (scaled) sum over all cosets of G of the sums above.

Theorem. Wedge is associative

$$(T\wedge S)\wedge R = T\wedge(S\wedge R).$$

Proof. We claim $(T\wedge S)\wedge R = Alt(T\otimes S\otimes R)$.
By definition,

$$(T\wedge S)\wedge R = Alt((T\wedge S)\otimes R)$$

so

$$(T\wedge S)\wedge R - Alt(T\otimes S\otimes R) = Alt((T\wedge S)\otimes R) - Alt(T\otimes S\otimes R)$$

$$= Alt((T\wedge S - T\otimes S)\otimes R).$$

by linearity of $\otimes$ and Alt.

Now

$$\mathrm{Alt}(T\wedge S - T\otimes S) = \mathrm{Alt}(T\wedge S) - \mathrm{Alt}(T\otimes S)$$
$$= T\wedge S - T\wedge S = 0$$

(since $T\wedge S$ is alternating). By Lemma, this implies

$$\mathrm{Alt}([T\wedge S - T\otimes S]\otimes R) = 0,$$

so

$$(T\wedge S)\wedge R = \mathrm{Alt}(T\otimes S\otimes R), \text{ as required.}$$

Examples. $V = \mathbb{R}^3$

$$T(\vec{v}) = v_1 + v_2, \qquad S(\vec{v}) = v_3$$

$$T\otimes S(\vec{v},\vec{w}) = (v_1+v_2)\cdot w_3$$

$$T\wedge S(\vec{v},\vec{w}) = (v_1+v_2)\cdot w_3 - (w_1+w_2)\cdot v_3$$

Our goal now is to derive a basis for $\Lambda^p(V^*)$, using the wedge product.

~~Claim.~~ If T is any p-tensor, we recall that a basis $\varphi_1, \dots, \varphi_k$ for V^* yields a basis for $\mathcal{J}^p$ ~~(V)~~ in the form

$$T = \sum t_{i_1,\dots,i_p} \phi_{i_1} \otimes \dots \otimes \phi_{i_p}$$

So if T altermates,

$$T = \mathrm{Alt}(T) = \sum t_{i_1,\dots,i_p} \mathrm{Alt}(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_p})$$

$$= \sum t_{i_1,\dots,i_p} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_p}.$$

So the set of p-tensors in the form $\varphi_{i_1} \wedge \dots \wedge \varphi_{i_p}$ span $\Lambda^p(V^*)$. We will see they are <u>not</u> independent.