Math 2500 – Lecture 4 – Lines and planes.

The equation of a line through $\vec{r}_0$ parallel to $\vec{v}$, $\vec{r}(t) = \vec{r}_0 + t$

Note: <u>parametric</u> equation with parameter t.

Examples. $\vec{r}_0 = (1,2)$ $\vec{v} = (1,1)$, line segment from $\vec{u}$ to $\vec{v}$.

Definition. A <u>linear combination</u> of $\vec{v}_1, \ldots, \vec{v}_n$ is any sum in the form $a_1\vec{v}_1 + \ldots + a_n\vec{v}$ where a_i are scalar

Proposition. Every vector in $\mathbb{R}^n$ is a linear combination of $\vec{e}_1 = (1, 0, \ldots, 0), \ldots, \vec{e}_n = (0, \ldots, 0, 1)$.

Examples. $\vec{i}, \vec{j}, \vec{k}$, plane <u>spanned</u> by $\vec{u}, \vec{v}$ (through origin)

"Parametric" equation for a plane through $\vec{p}_0$ parallel to plane spanned by $\vec{u}$ and $\vec{v}$: $\vec{r}(s,t) = \vec{p}_0 + s\vec{u} + t\vec{v}$.

Definition. The vector $\vec{n}$ is <u>normal</u> to the plane P though $\vec{p}$ iff $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$ for every $\vec{p} \in P$.

Implicit equation for a plane: $ax + by + cz = d$. (in $\mathbb{R}$

Example. $\vec{n} = (3,4,5)$, $\vec{p}_0 = (1,1,1)$

Proposition. P is spanned by $\vec{u}, \vec{v} \iff \vec{u} \times \vec{v}$ is normal

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Classical examples.

- Line of intersection of 2 planes.
- Distance from point to plane.
- Distance from point to line.
- Distance between lines.

Challenge question. Distance from point to line <u>segment</u>.

Math 2500 - Lecture 5 - Vector valued functions & Derivatives.

Definition. A <u>parametrized curve</u> is a function
$$\vec{v}(t) = (v_1(t), \ldots, v_n(t)).$$

Examples. circle, helix, cycloid, (mathematica demo).

Definition. The <u>tangent vector</u> to $\vec{v}(t)$, is
$$\vec{v}'(t) = (v_1'(t), \ldots, v_n'(t)).$$
If t = time, $\vec{v}'(t)$ = <u>velocity vector</u>.

Examples. circle, helix, cycloid.

Definition. The ~~curvature vector~~ second derivative of $\vec{v}(t)$ is
$$\vec{v}''(t) = (v_1''(t), \ldots, v_n''(t)).$$
If t = time, $\vec{v}''(t)$ = <u>acceleration vector</u>.

Definition. If t = time, <u>speed</u> = $\|\vec{v}'(t)\|$, acceleration = $\|\vec{v}''(t)\|$.

Rules for vector-valued derivatives.

$$\frac{d}{dt}\vec{C} = \vec{0}$$

$$\frac{d}{dt} c\,\vec{u}(t) = c\,\vec{u}'(t)$$

$$\frac{d}{dt} c(t)\,\vec{u}(t) = c'(t)\,\vec{u}(t) + c(t)\,\vec{u}'(t).$$

$$\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}'(t) + \vec{v}'(t).$$

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t).$$

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derivative of curve on sphere.

Math 2500 - Lecture 6 - Arc Length and Curvature.

Idea: distance = integral of speed w.r.t. time.

Definition. The length of the portion of $\vec{r}(t)$ between $t=a$ and $t=b$ is $\int_a^b \|\vec{r}\,'(t)\|\,dt$.

Example. helix.

Definition. The <u>unit</u> <u>tangent</u> <u>vector</u> $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.

Example. helix.

~~Definition. The curvature of $\vec{r}(t)$~~

Definition. $\vec{r}(t)$ is <u>parametrized</u> <u>by</u> <u>arclength</u> when $\vec{T}(t) = \vec{r}\,'(t)$. (In this case, we usually use s instead of t for parameter).

It is true that $ds/dt = \|\vec{r}\,'(t)\|$, $dt/ds = \dfrac{1}{\|\vec{r}\,'(t)\|}$.
be careful here

Example. helix.

Definition. The <u>curvature</u> of $\vec{r}(t)$ is $\left\| \dfrac{d\vec{T}}{ds} \right\| = \kappa$

Proposition. $\kappa(t) = \dfrac{1}{\|\vec{r}\,'(t)\|} \left\| \dfrac{d\vec{T}}{dt} \right\|$.

Example. curvature of parabola, (mathematica demo.)

$$\frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}$$

Math 2500 - Lecture 6 - ~~Partial Derivatives~~ Functions on $\mathbb{R}^n$

Definition. Let $D \subseteq \mathbb{R}^n$. A function $f: D \to \mathbb{R}$ is a rule which assigns a value $f(\vec{x}) \in \mathbb{R}$ to each $\vec{x} \in D$.
A function $F: D \to \mathbb{R}^m$ is a collection of functions

$$F(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x})), \quad f_i: D \to \mathbb{R}.$$

Examples. $f(\vec{x}) = x_1 x_2 \sin(x_3) + 3$. $\quad F(\vec{x}) = (x_1 x_2, x_1 + x_3, 3)$.

$f: \mathbb{R}^3 \to \mathbb{R} \qquad F: \mathbb{R}^3 \to \mathbb{R}^3$

Domain and range. $f(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$.

Level curves. Graphs. Level surfaces. Mathematica demos

Definition. Given $\vec{x}, \vec{y} \in \mathbb{R}^n$, we let $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$.

Definition. Given $F: \mathbb{R}^n \to \mathbb{R}^m$, we say $\lim_{\vec{x} \to \vec{x}_0} F(\vec{x}) = \vec{L}$ if for each $\epsilon > 0$ $\exists$ a corresponding $\delta > 0$ so that if $d(\vec{x}, \vec{x}_0) < \delta$ then $\delta(F(\vec{x}), \vec{L}) < \epsilon$.

Examples. Multivariable mathematica demo. (!)
Definition. Continuous function.

Bad Example. $f(x,y) = 2x^2y / (x^4 + y^2)$.

Definition. A set $D \subset \mathbb{R}^n$ is ~~closed~~ <u>open</u> if for every $\vec{x} \in D$ there is some $\epsilon > 0$ so that if $d(\vec{x}, \vec{y}) < \epsilon$, $\vec{y} \in D$.
A set $D \subset \mathbb{R}^n$ is <u>closed</u> if $\mathbb{R}^n - D$ is open.

Examples. Inequalities. Pictures. Sets are not doors.

Definition. A set D is <u>bounded</u> if $\exists$ some N so that $d(\vec{x}, \vec{0}) < N$ for all $\vec{x} \in D$.

Theorem. A continuous function $f: D \to \mathbb{R}$ on a closed, bounded D has a max and min.

Math 2500 - Lecture 1 - Partial Derivatives.

~~Definit~~

Definition. The partial derivative $\frac{\partial f}{\partial x_i}$ of the function $f: D \subset \mathbb{R}^n \to \mathbb{R}$ is the derivative of the function $g(x) = f(x_1, x_2, \dots, x, x_{i+1}, \dots$ with respect to x.

Examples. $f(x,y) = y \sin x$, $f(x_1, \dots, x_n) = x_1^2 + x_2^3 + \dots + x_n^{n+1}$.

Notation. $\frac{\partial f}{\partial x_i}(\vec{x}_0)$, $f_{x_i}(\vec{x}_0)$, $\frac{\partial f}{\partial x}\Big|_{\{\vec{x}_0\}}$, f_x, $\frac{\partial f}{\partial x}$.

Higher order partials. Definition of $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$.

Theorem. If $f(x,y)$, f_x, f_y, f_{xy}, f_{yx} are continuous in an open set around (x_0, y_0) then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

Examples. $\cos^2(3x - y^2)$, $x \sin y + e^y$, $x \ln xy$.

Definition.

$$\Delta f(\vec{x}_0, \vec{\Delta x}) = f(\vec{x}_0 + \vec{\Delta x}) - f(\vec{x}_0)$$

Definition. A function $f: D \to \mathbb{R}$ is *differentiable* at $\vec{x}$ if

$$\Delta f = \left(f_{x_1}(\vec{x}_0)\right)(\vec{\Delta x})_1 + \dots + \left(f_{x_n}(\vec{x})\right)(\vec{\Delta x})_n + \epsilon_1 (\vec{\Delta x})_1 + \dots + \epsilon_n (\vec{\Delta x})_n.$$

where $\epsilon_1, \dots, \epsilon_n \to 0$ as $\vec{\Delta x} \to \vec{0}$.

Theorem. If all partials of f are cts on an open D, then f is differentiable at each $\vec{x} \in D$.

Theorem. If f is differentiable at $\vec{x}$, f is cts at $\vec{x}$.

More computational examples. Applications! Lunocet.