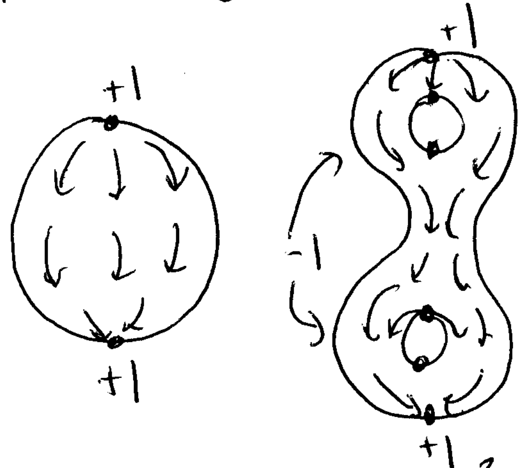
Larger Lefschetz Numbers

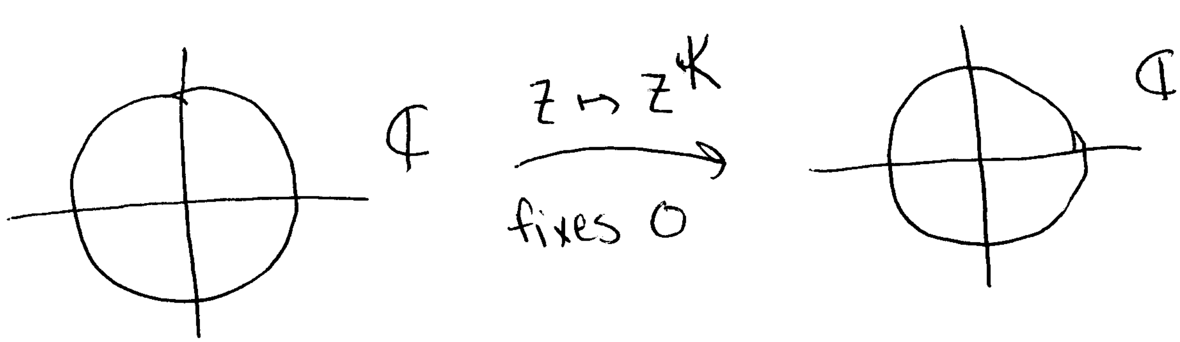
We ended last class with a discussion of fields on surfaces, and their associated fixed points



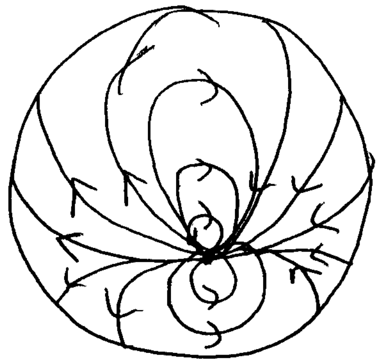
We used this to compute $\chi(S)$ for surfaces in $\mathbb{R}^3$.

Theorem. If S is a (compact, smooth, orientable) surface of genus g, then $\chi(S) = 2-2g$.

What about other fixed points?



Or



?

These fixed points are not Lefschetz! But by our previous prop, a homotopic map has only Lefschetz fixed points.

Splitting Theorem. Let U be a neighborhood of a fixed point x which contains no other fixed points of f. $\exists$ a homotopy f_t of f which $\equiv f$ outside U so that f_1 has only Lefschetz fixed points and each $f_t = f$ outside some compact subset of U.

We now want to define:

Definition. If x is an isolated fixed point of $f: \mathbb{R}^k \to \mathbb{R}^k$, we let f_1 be a local homotopy with only Lefschetz fixed points and

$$L_x(f) = \sum_{f_1(x)=x} L_x(f_1)$$

Here is a surprising idea: On $\mathbb{R}^k$, we can define a map

$$g_x(z) = \frac{f(z)-z}{|f(z)-z|}$$

near any isolated fixed point z of f. ~~Define~~ Take a sphere S^{k-1} centered at z and consider

$$g_x : S^k \to S^k$$

Definition. $L_x(f) = \deg g$.

Proof.

Assume $f: U \subseteq \mathbb{R}^k \to \mathbb{R}^k$ fixes only $\vec{0}$ ~~in~~ in U.
Now choose a bump function φ so that

φ is smooth
is 1 in a neighborhood V of 0 contained in a compact subset K of U
is 0 outside K

Our idea is to choose some vector $\vec{v}$ so that

$$f_t(x) = f(x) + t\,\varphi(x)\,\vec{v}$$

works.

Claim. If $\vec{v}$ is really small, then f_t has no fixed points outside V, in U.

Consider $K-V$. This is compact, and f has no fixed points on it, since $K-V \subset U$, and $0 \notin K-V$. So $|f(x)-x| > c > 0$ on $K-V$.
Choose $|\vec{v}| < c/2$. Since $t\varphi \leq 1$, this works.
Outside K, $f_t = f$, so there are no fixed points.

By Sard's theorem, we can pick $\vec{v}$ close to 0 so that $-\vec{v}$ is a regular value for

$$x \mapsto f(x) - x.$$

Now suppose x is a fixed point of f_1. We know $x \in V$, so

$$f_1(x) = f(x) + \vec{v}.$$

Thus

$$d(f_1)_x = df_x.$$

Now x is Lefschetz for $f_1 \iff d(f_1)_x - I$ nonsingular. But

$$d(f_1)_x - I = df_x - I$$

$$= d(f(x) - x)_x = \text{nonsingular, since}$$

$f_1(x) = x \Rightarrow f(x) - x = -\vec{v}$ and $-\vec{v}$ is a regular value for $f(x) - x$.

Now in general, take local coordinates.

These two ideas agree!

Example. Lefschetz examples, do calculations.

Proposition. At <u>Lefschetz</u> fixed points, the numbers agree.

Proof. As usual, let $X = \mathbb{R}^n$ and suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ has 0 as a <u>Lefschetz</u> fixed point.

We can write

$$f(x) = Ax + \epsilon(x), \quad \text{where } A = df_0$$

near the origin. Since $A - I$ is an isomorphism, it maps the unit sphere to an ellipsoid at least $c > 0$ from 0. By linearity,

$$|(A-I)z| > |z|c$$ ~~from origin~~.

But ~~it~~ then choosing B around $\vec{0}$ small enough so $\frac{|\epsilon(z)|}{|z|} < c/2$ ($\epsilon(z)$ is $\underline{\text{2nd}}$ order in z), we see that on B, if ~~$f_t = Az + t\epsilon(z)$,~~

$$\cancel{|f_t(z) - z| = |Az + t\epsilon(z) - z}$$

$$f_t(z) = Az + t\epsilon(z),$$

then for all t in $[0,1]$,

$$|f_t(z) - z| = |Az + t\epsilon(z) - z|$$

$$\cancel{\geq |Az| - |\epsilon(z)| - |z|.}$$

~~So~~

$$\geq |Az - z| - t|\epsilon(z)|$$

~~But $|Az| > c|z|$, $|\epsilon(z)| < c|z|/2$, so for small~~

$$\geq c|z| - c|z|/2 > 0$$

So we have $|f_t(z) - z| > 0$.

This means that f_t ~~is~~ defines a homotopy ~~between maps~~ of maps

$$g_t(z) = \frac{f_t(z) - z}{|f_t(z) - z|}$$

defined on ~~B~~ a small S^{n-1} around $\vec{0}$ to S^{n-1}. But then

$$\begin{aligned} \deg g_1(z) &= \text{our def. of Lefschetz degree} \\ \| \\ \deg g_0(z) &= \deg \frac{(A-I)z}{|(A-I)z|}. \end{aligned}$$

Fact. ~~A~~ A linear isomorphism $B : \mathbb{R}^n \to \mathbb{R}^n$ is homotopic to I, if $\det B > 0$, through isomorphisms

and to a reflection if $\det B < 0$.

In the first case, $\deg \frac{A-Iz}{|A-Iz|} = \deg \frac{z}{|z|} = 1$,

in the second, $\deg \frac{(-z_1, z_2, \ldots, z_n)}{|z|} = -1$.

Cool! We can now define Lefschetz number again in a more insightful way.

Definition. Let $f: X \to X$ be a map with finitely many fixed points x_i on a compact manifold

$$L(f) = \sum_{x_i} L_{x_i}(f),$$

where

$$L_{x_i}(f) = \deg \frac{f(x)-x}{|f(x)-x|} : \text{Ball around } x_i \to S^{\dim X - 1}.$$

Try some examples!!

Lefschetz number 2?