Lefschetz Fixed Point Theory

Consider any map $f : X \to X$. We see

$$\text{graph } f = \operatorname{Im} I \times f : X \to X \times X$$

is a map from a space of dimension $\dim X$ to a space of dimension $2 \dim X$. Thus we can define

$$I(\text{graph } f, \Delta) = \text{a count of points where } (x, f(x)) = (x, x) \text{ or } x = f(x)$$

This should detect fixed points!

Definition. $I(\text{graph } f, \Delta) = L(f)$ is the <u>Lefschetz</u> <u>number</u> of f.

Proposition. If $L(f) \neq 0$, f has a fixed point.

Proposition. $L(f)$ is a homotopy invariant of f.

Note that if $f = I$, $L(f) = \chi(X)$.

So

Proposition. If $f \simeq I$, then $L(f) = \chi(X)$. If $\exists$ a map $f: X \to X$ homotopic to I with no fixed points, then $\chi(X) = 0$.

Example. $T^n = S^1 \times \cdots \times S^1$ has $\chi(T^n) = 0$.
Every map $f: S^2 \to S^2$ homotopic to the identity <u>has</u> a fixed point. (Hairy ball theorem)

The obvious maps to study are maps with graph $f \pitchfork \Delta$. These are called Lefschetz maps.

Lemma. If f is Lefschetz, f has finitely many fixed points.

Proposition. Every map $f: X \to X$ is homotopic to a Lefschetz map.

Proof. Technical, easy.

If f is Lefschetz, at each fixed point

$$T_x(\text{graph} f) + T_x(\Delta) = T_x(X \times X)$$

But

$$T_x(\text{graph} f) = \text{graph } df_x$$

$$T_x(\Delta) = \text{the diagonal } \Delta_x \text{ of } T_x X \times T_x X = T_x(X \times X)$$

so really this means

$$\text{graph } df_x + \Delta_x = T_x(X) \times T_x(X).$$

Counting dimensions, this holds $\Leftrightarrow$

$$\text{graph } df_x \cap \Delta_x = \vec{0}$$

But $\text{graph } df_x = \{(\vec{x}, df(\vec{x}))\}$, so this holds $\Leftrightarrow \nexists\ \vec{x} \in T_x X$ with $df(\vec{x}) = \vec{x}$, or

f is Lefschetz (at x) $\Leftrightarrow$ df_x does not have $+1$ as an eigenvalue

We define $L_x(f)$ to be the local intersection # of Δ, graph f at a fixed point, so

$$L(f) = \sum_{f(x)=x} L_x(f).$$

Proposition. If f is Lefschetz, then at any fixed point x, $df_x - I$ is an isomorphism of T_xX. Further

$$L_x(f) = +1 \iff df_x - I \text{ is orientation } \underline{\text{preserving}}$$

Proof. Let $A = df_x$, $\beta = \{v_1, \dots, v_k\}$ be a positive basis for T_xX. Int # is

$$\text{sign}\{(v_1,v_1), \dots, (v_k,v_k), (v_1,Av_1), \dots, (v_k,Av_k)\}$$

using linear operations, reduce to

$$\{\beta \times 0, 0 \times (A-I)\beta\}.$$

The 2-dimensional case.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $A = df_0$, so $f(x) = Ax + \epsilon(x)$.

If A has two independent eigenvectors,

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

in the right coordinates.

So

$$L_0(f) = \operatorname{sign}(\alpha_1 - 1)\operatorname{sign}(\alpha_2 - 1).$$