MATH 2300H

Final Exam

December 13, 2004

NAME (please print legibly): _____

Your University ID Number: _____

Please complete all 20 questions in the space provided. You may use the backs of the pages for extra space, or ask me for more paper if needed. Work carefully, and try to complete the problems you find easier before going back to the harder ones. Good luck!

QUESTION	VALUE	SCORE
1	10	
2	10	
3	10	
4	15	
5	10	
6	10	
7	15	
8	20	
9	10	
10	10	
11	10	
12	10	
13	10	
14	20	
15	10	
16	10	
17	10	
TOTAL	200	

1. (10 points) This question has three parts:

1. State the quotient law **for limits**.
   (Remember that it is different from the quotient rule for derivatives.)

2. State the product law **for limits**.
   (Again, remember that it is different from the product rule for derivatives.)

3. If it exists, evaluate the limit
$$\lim_{x \to 4} \frac{x^2 - 16}{2 - \sqrt{x}}.$$

ANSWER: _____

2. (10 points) If it exists, find the one-sided limit

$$\lim_{x \to 3^+} \frac{\sqrt{x^2 - 6x + 9}}{x - 3}.$$

ANSWER: _____

3. (10 points) This question has two parts:

1. Define **continuity at a point** for a function $f(x)$ at a point a.

2. State the **intermediate value theorem** for continuous functions on a closed interval.

4. (15 points) This question has three parts:

1. Write down the **definition of the derivative as a limit** for a function $f(x)$ at a point a.

2. Use that definition (and the limit laws, but *not* L'Hopital's rule) to find the derivative of

$$f(x) = \frac{x+1}{x-1}.$$

ANSWER: _____

3. Compute $f'(x)$ using the quotient rule for derivatives (and check the result against your answer in part 2).

ANSWER: _____

5. (10 points) Compute the derivative of

$$f(t) = (t^2 + 1)(t^3 + t^2 + 1) \tag{1}$$

using the product rule.

ANSWER: _____

6. (10 points) This question has two parts:

1. State **the chain rule** for the derivative of the composition $f(g(x))$ of functions $f(x)$ and $g(x)$.

2. Use the chain rule to find the derivative of $h(x) = \sin(x^3)\cos(x^3)$. Be sure to state explicitly the "outer" function $f(x)$ and the "inner" function $g(x)$.

ANSWER: _____

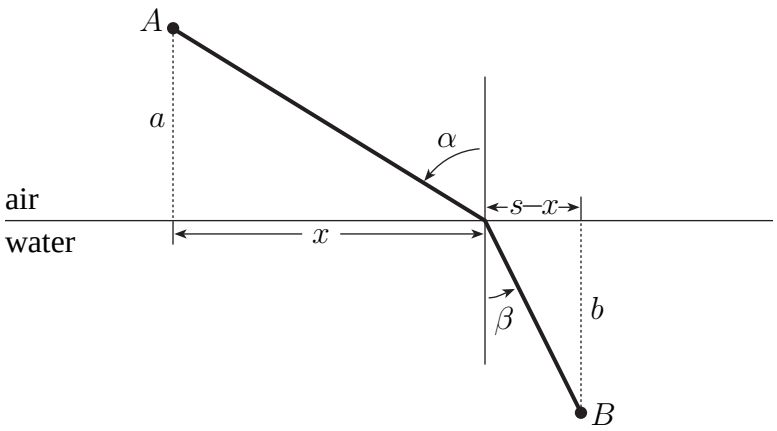
7. (15 points) Find the **maximum** and **minimum** values of the function

$$f(x) = x + \frac{4}{x}$$

on the closed interval $[1, 4]$. Note: we are asking for the *values* (or "y values") of the max and min instead of their *locations* (or "x values").

ANSWER: _____

8. (20 points) As we learned in the last lectures, light travels at speed c in air and a slower speed v in water. The ratio c/v is about $4/3$. Suppose that a light source is located at point A on the diagram below, while an observer is located at point B. **Fermat's principle** tells us that the light moves along the path from A to B which takes the least time.



1. Write down the time $T(x)$ required for light to complete the path shown below in terms of the **variable** x and the **constants** a, b, c, v, and s.

ANSWER: _____

2. Prove **Snell's law**

$$\frac{\sin\alpha}{\sin\beta} = \frac{c}{v} \tag{2}$$

by minimizing the function $T(x)$ that you derived in part 1.

9. (10 points) Using implicit differentiation, find the equation of the tangent line to the curve

$$xy = 6e^{2x-3y}$$

at the point $(3, 2)$. Please express your answer in point-slope form.

ANSWER: _____

10. (10 points) This problem has two parts:

1. Fill in the blank to write down the **linear approximation** to a function $f(x)$ at a point a.

   $$f(x) \approx$$

2. Use linear approximation to estimate the numerical value of $e^{1/10}$.

ANSWER: _____

11. (10 points) The width of a rectangle is half its length. At what rate is the **area** of the rectangle increasing if the width is 10 cm and the width is increasing at 0.5 cm/sec?

ANSWER: _____

12. (10 points) Find the intervals on which the function

$$f(x) = \frac{x}{x+1}$$

is **increasing** and **decreasing**.

ANSWER: _____

13. (10 points) Find all the critical points of

$$f(x) = xe^{-2x}$$

and classify each as a **local maximum**, **local minimum**, or **saddle point**.

ANSWER: _____

14. (20 points) Sketch the best graph you can of

$$f(x) = \frac{1}{e^x + e^{-x}}.$$

Be sure to indicate

1. Horizontal and vertical asymptotes.
2. Critical points and increasing/decreasing intervals.
3. Intervals on which the graph is concave up/concave down.

ANSWER: _____

15. (10 points) This problem has two parts:

1. State **L'Hopital's rule**.

2. Use L'Hopital's rule to compute
$$\lim_{x\to 0} \frac{\ln(1+x)}{x}.$$

ANSWER: _____

3. (5 bonus points) Compute
$$\lim_{x\to \infty} \frac{2^x}{3^x}.$$

ANSWER: _____

16. (10 points) This problem has two parts:

1. Compute
$$\int (5\cos(10x) - 10\sin(5x))\,\mathrm{d}x.$$

ANSWER: _____

2. Differentiate your answer above. Do you get the expected answer?

ANSWER: _____

17. (10 points) Solve the **initial value problem**

$$\frac{dy}{dx} = \frac{x}{y}, \qquad y(3) = 5.$$

for the function $y(x)$.

ANSWER: _____