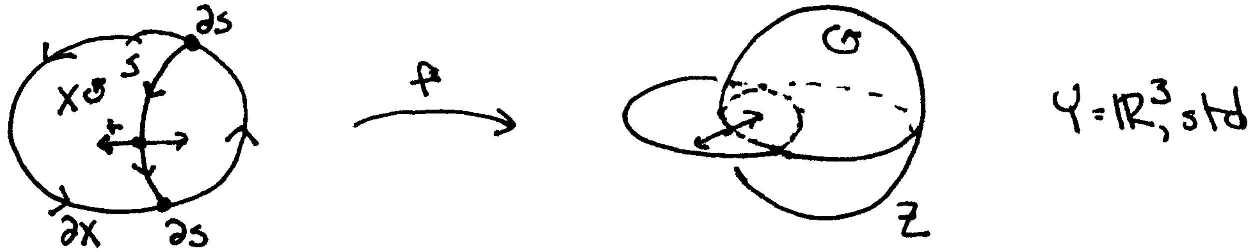
A weird property of the preimage orientation

Let's return to our previous setup

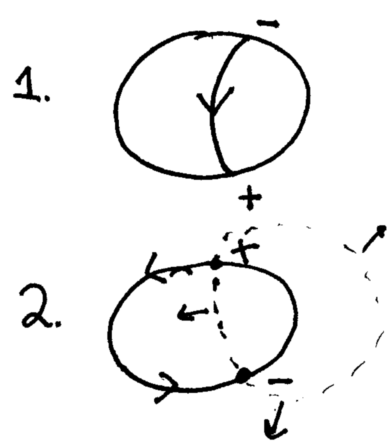


We can orient ∂S in two ways:

1. with the boundary orientation of the preimage orientation of $S = f^{-1}(Z)$.

2. with the preimage orientation under ∂f.

In our example, let's check these possibilities.



they disagree!

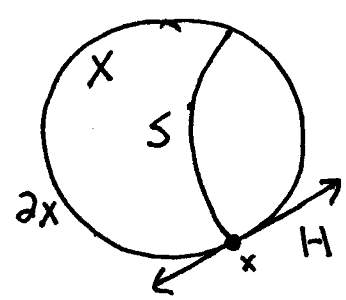
Proposition. $\partial[f^{-1}(Z)] = (-1)^{\operatorname{cod} Z} (\partial f)^{-1}(Z)$.

The statement illustrates that from now on, when we are dealing with oriented mflds, we will freely use

"∂" to mean "boundary with boundary orientation"

"f^{-1}" to mean "preimage with preimage orientation"

Proof.



Choose H in $T_x(\partial X)$ so

$$H + T_x(\partial S) = T_x(\partial X)$$

and H is the orthogonal complement of $T_x(\partial S)$ in $T_x(\partial X)$.

Now we claim H is complementary to $T_x(S)$ in $T_x X$.

1. $H \cap T_x S = H \cap (T_x S \cap T_x \partial X)$ since $H \subset T_x(\partial X)$

$= H \cap (T_x(\partial S)) = 0.$

2. $\dim H + \dim \partial S = \dim \partial X$

$\dim H + \dim S - 1 = \dim X - 1$

$\dim H + \dim S = \dim X$

so $H + T_x S$ is top dimensional and must be $T_x X$.

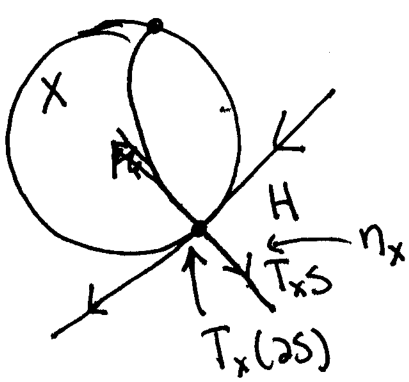
We will use the same H to define the preimage orientation on both ∂S and S.

(Note that we used comment that $N_x(S; X)$ need not be the orthogonal complement — another complement would do as well.)

Now

df_x and $d(\partial f)_x$ agree on H,

so if we orient the image of H in $T_z Y$ using the orientations on Z, Y, this pulls back to the <u>same</u> orientation on H itself.



Let n_x be the outward normal to ∂S in $T_x S$. We observe that the ∂ orientation on $T_x(\partial X)$ is given by

$$\mathrm{span}(n_x) + T_x(\partial X) = T_x X.$$

(We need only show that n_x is an outward pointing normal to X, as well. This follows from the fact that $S \pitchfork \partial X$, which follows in turn from our assumptions $f \pitchfork Z$, $\partial f \pitchfork Z$.

~~Proof. Suppose $S \not\pitchfork \partial X$.~~

But how? Should add proof.)

Now $T_x X$ is part of the preimage orientation of S <u>and</u> ∂S, so we have (oriented)

$$H + T_x S = \mathrm{span}(n_x) + \underbrace{H + T_x(\partial S)}_{\text{preimage orientation}}.$$

using dim H transpositions, we see

$$= (-1)^{\dim H} H + \underbrace{\mathrm{span}(n_x) + T_x(\partial S)}_{\text{boundary orientation of } \partial S}$$

Since H has the same orientation on both sides, we see that given a (boundary) positive basis β for

~~orientation $T_x\partial S$~~

$$\cancel{T_x S = (-1)^{\dim H} \mathrm{span}(n_x) + T_x(\partial S)}$$

$T_x(\partial S)$, the corresponding basis (n_x, β) for $T_x S$ has sign $(-1)^{\dim H}$ as a (preimage) basis for $T_x S$.

Thus the two orientations are related by $(-1)^{\dim H}$. But

$$\dim H = \operatorname{codim} S = \operatorname{codim} Z,$$

so we're done.