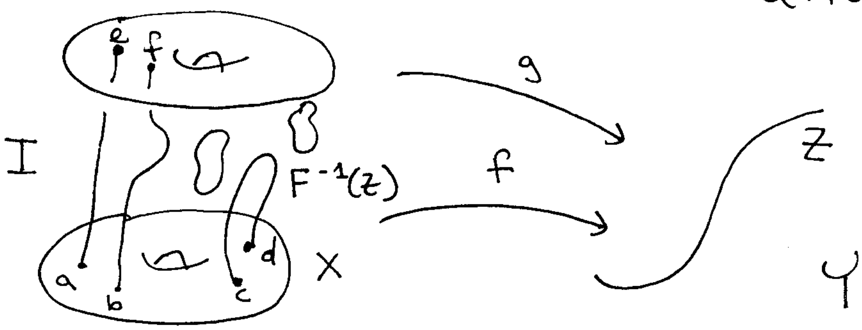
Orientation and Intersection #.

We have now studied the situation



where Z and X have complementary dimension.

We ~~need to~~ recall the setup we used to show that

$$\#(f^{-1}(Z)) = \#(g^{-1}(Z)) \mod 2$$

for homotopic maps $f, g : X \to Y$, each $\pitchfork Z$.

By the extension theorem, we could assume that the homotopy $F : X \times I \to Y$ was $\pitchfork Z$. Then

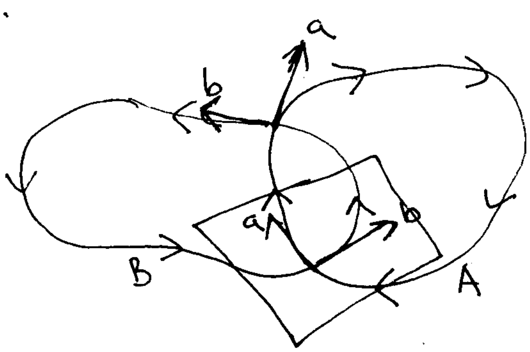
$F^{-1}(Z)$ is a 1-manifold with boundary
$$f^{-1}(Z) \sqcup g^{-1}(Z),$$

and the boundary of any 1-manifold has an even number of points.

In our example drawing, $f^{-1}(z)$ contains four points – a, b, c, d. But of these, only a, b should "count" as they extend to points in $g^{-1}(z)$. The other two should "cancel".

Goal: Define a counting system that allows us to distinguish between these cases.

Idea.



"orient" A, B, and $\mathbb{R}^2$ by choosing a ctsly varying preferred basis for each tangent space.

At each intersection consider

sgn det A where A maps the basis $\{a, b\}$ to the preferred basis for $T_p\mathbb{R}^2$

to be the "sign" of the crossing. Add up signs of crossings to get intersection #.

Orientation of vector spaces.

If V is a vector space (finite dim'l, real), let $\beta = \{v_1, \dots, v_n\}$, $\beta' = \{v_1', \dots, v_n'\}$ be bases for V. Such a pair of bases has a unique linear transformation so $\beta' = A\beta$.

Definition. β and β' have <u>the same orientation</u> if $\beta' = A\beta$ and $\det A > 0$.

Lemma. orientation ~~defin~~ is an equivalence relation on bases for V.

Proof. reflexive $\beta = I\beta$, $\det I > 0$.
symmetric $\beta' = A\beta \Rightarrow A^{-1}\beta' = \beta$. $\det(A)\det(A^{-1}) = \det(I) > 0$. So both $\det(A)$ and $\det(A^{-1})$ are positive if either is.
transitive

$\beta' = A\beta$, $\beta'' = A'\beta'$.

$\beta'' = (A'A)\beta$ and again

$\det(A'A) = \det(A')\det(A) > 0$ if $\det(A)$, $\det$

Definition. An <u>orientation</u> of V is a decision to call bases in one equivalence class <u>positive</u> and bases in other equivalence classes <u>negative</u>.

We note that:

Lemma. There are two equivalence classes of bases of V.

Proof. Exercise.

Example. $\{e_1, e_2, e_3\}$ is positively oriented in $\mathbb{R}^3$.

$\{e_2, e_1, e_3\}$ is

$\{-e_1, e_2, e_3\}$ is   oriented

$\{e_2, e_3, e_1\}$ is

Definition. A 0-dimensional vector space is oriented by choosing a + or - sign for the "empty basis".

Suppose that $A: V \to W$ is an isomorphism. Then If $\beta' = M\beta$ in V, $A\beta' = (AM)\beta = (AMA^{-1})A\beta$ in W. But $\det(AMA^{-1})$ has the same sign as $\det(M)$, so $\beta' \cong \beta \iff A\beta' \cong A\beta$.

Lemma. An isomorphism of vector spaces either <u>preserves</u> $(+ \to +)$ or <u>reverses</u> $(+ \to -)$ orientation.

Orientation of manifolds.

Definition. The standard orientation on $\mathbb{R}^n$ assigns $+$ to the basis $\{e_1, \dots, e_n\}$. The standard orientation of $\mathbb{R}^n$ as a manifold assigns $+$ to this basis for every tangent space.

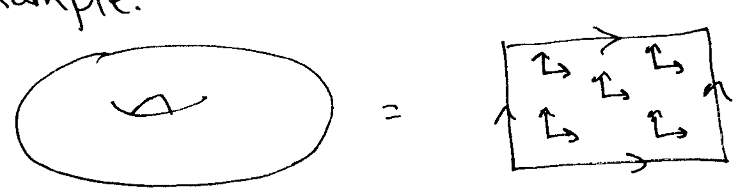
We can now define orientation for manifolds:

Definition. An <u>orientation for the manifold X</u> is a choice of orientation for each $T_x X$ so that around each $p \in X$ $\exists$ a set of local coordinates

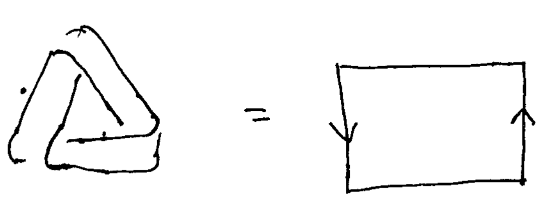
$$h : U \subset H^k \to X$$
$$H^k \subset \mathbb{R}^k$$

so that $\cancel{dh}$ is $\mathbb{R}^k$ orientation preserving for all $p \in U$.
$dh : T_p \mathbb{R}^k \to T_{h(p)} X$

Example.



T^2 oriented by $\{d\theta, d\varphi\}$



not orientable.

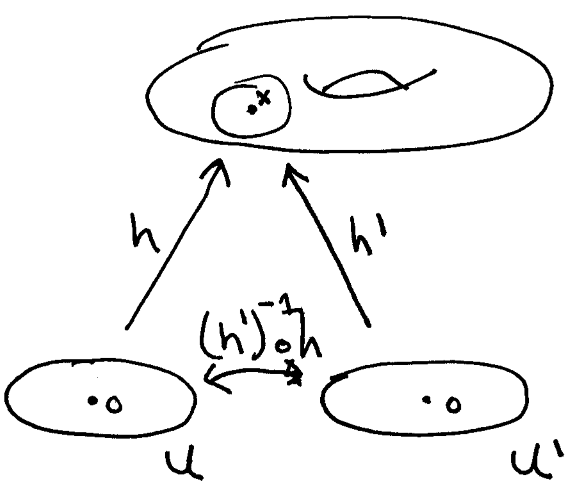
We now show

Proposition. A connected orientable manifold has exactly two orientations.

Proof. Given two orientations of X, we want to show that either they agree everywhere (they are the same) or they disagree everywhere (they are opposite).

We show that the set of points on which they agree is closed and open.

Claim 1. (Open). Suppose at $x \in X$ we have two sets of local coordinates h and h' which preserve the two orientations.



Then

$$d((h')^{-1} \circ h) : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$$

has positive determinant since the orientations agree.

(Why? This is a good point to explain in class.)

By continuity of $d((h')^{-1} \circ h)_y$ in y and continuity of det, this map has <u>positive</u> determinant in a neighborhood ~~&~~ V of 0. The orientations agree on $h'(V)$, an open neighborhood of x.

Claim 2. (closed) The same, substituting "disagree" and "negative determinant".

Definition. A manifold X with an orientation if called an <u>oriented manifold</u>. The ~~ma~~ oriented manifold obtained by reversing the orientation is called $-X$.

Orienting product manifolds

An orientation is a way to assign a sign to every basis for every tangent space of X. We can define an orientation by assigning a sign to only one basis (smoothly) and using previous proposition.

Suppose X, Y are oriented and only ~~two~~ one has a boundary. Then $X \times Y$ is a manifold with boundary.

Given $(x,y) \in X \times Y$,

$$T_{(x,y)}(X \times Y) = T_x X \times T_y Y.$$

If $\alpha = \{v_1, \ldots, v_k\}$, $\beta = \{w_1, \ldots, w_l\}$ are bases for $T_x X, T_y Y$ then

$$(\alpha \times 0, 0 \times \beta) = \{(v_1, 0), \ldots, (v_k, 0), (0, w_1), \ldots, (0, w_l)\}.$$

we let

$$\operatorname{sign}(\alpha \times 0, 0 \times \beta) = \operatorname{sign}(\alpha)\operatorname{sign}(\beta).$$

Lemma. This assignment defines an orientation on ~~X~~ $X \times Y$.

Proof. We have to check that the definition doesn't depend on α, β. So suppose

$$(\alpha' \times 0, 0 \times \beta') = A(\alpha \times 0, 0 \times \beta).$$

Then A is in block form

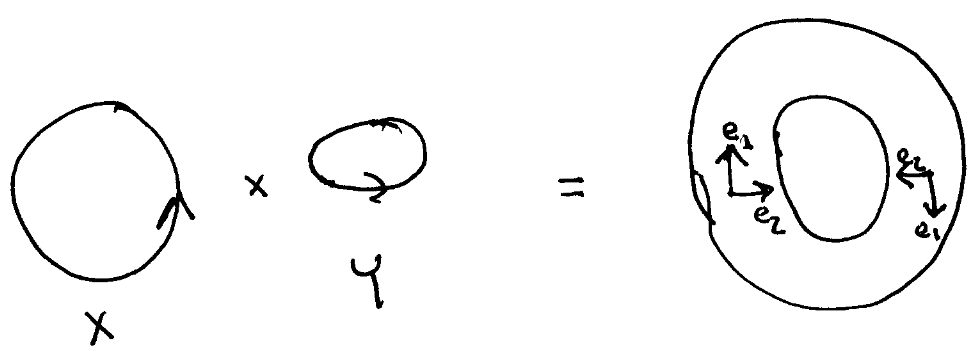
$$A = \begin{bmatrix} A_\alpha & 0 \\ 0 & A_\beta \end{bmatrix}, \quad \text{with} \quad \begin{array}{l} \alpha' = A_\alpha \alpha \\ \beta' = A_\beta \beta \end{array}$$

And

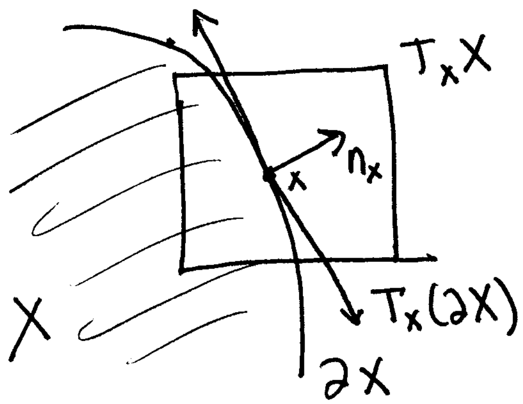
$$\det A = \det A_\alpha \det A_\beta$$

by rules for determinants.

Picture:



Orienting the boundary of a manifold.



We see:

$$T_x(\partial X) \subset T_x X$$

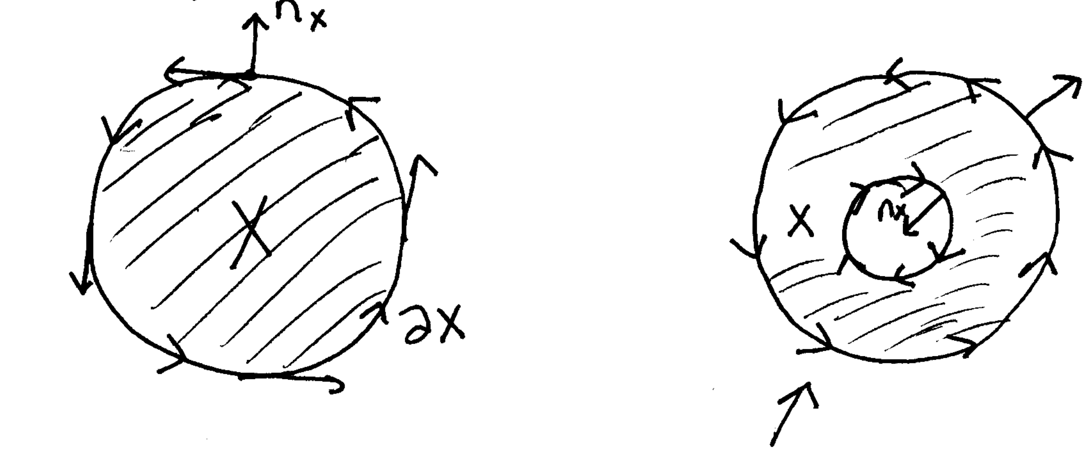
is a hyperplane. It has two normal vectors, one pointing into X the other pointing out of X. We call the <u>outward normal</u> n_x.

Given an orientation for X, and a basis β for $T_x(\partial X)$, the <u>boundary orientation</u> on ∂X sets

$$\text{sign } \beta = \text{sign}\{n_x, \beta\}$$

(By a similar argument about matrices in block form, this definition is consistent with the equivalence relation on bases for $T_x(\partial X)$.)

Example. Suppose X has std. orientation in $\mathbb{R}^2$



notice! boundary orientation on circles differ from one another

Exploring the example, we observe that for any homotopy $I \times X$, each X_t has an orientation from X_0.