Borsuk-Ulam (and consequences)

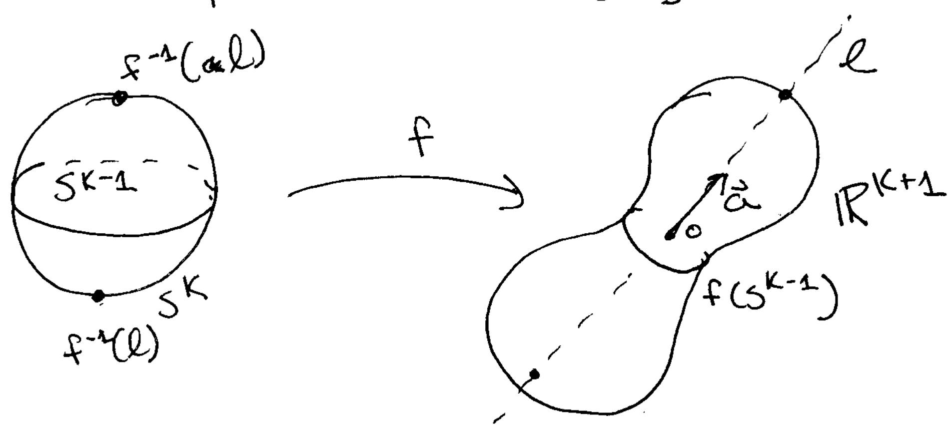
Borsuk-Ulam Theorem.

Let $f: S^k \to \mathbb{R}^{k+1}$ be a smooth map which avoids the origin and has

$$f(-x) = f(x) \text{ for all } x \in S^k.$$

Then $w_2(f, 0) = 1$.

Proof. We will prove it for $k = 1$ in homework. So assume the theorem for $k-1$, and take a map $f: S^k \to \mathbb{R}^{k+1} - \{0\}$.



Let $g: S^{k-1} \to \mathbb{R}^{k+1} - \{0\}$ be the restriction of f. Consider the maps

$$\frac{g}{|g|}: S^{k-1} \to S^k \qquad \frac{f}{|f|}: S^k \to S^k$$

and choose $\vec{a} \in S^k$ so that $\vec{a}$ is a regular value for both maps (Sard).

This means that $\vec{a} \notin \mathrm{Im}\left(\frac{g}{|g|}\right)$ (the dimension of S^{k-1} is too low) and that if the line ℓ is $\mathrm{span}(\vec{a}) \subset \mathbb{R}^{k+1}$, then $f \pitchfork \ell$, $g(S^k) \cap \ell = \emptyset$.

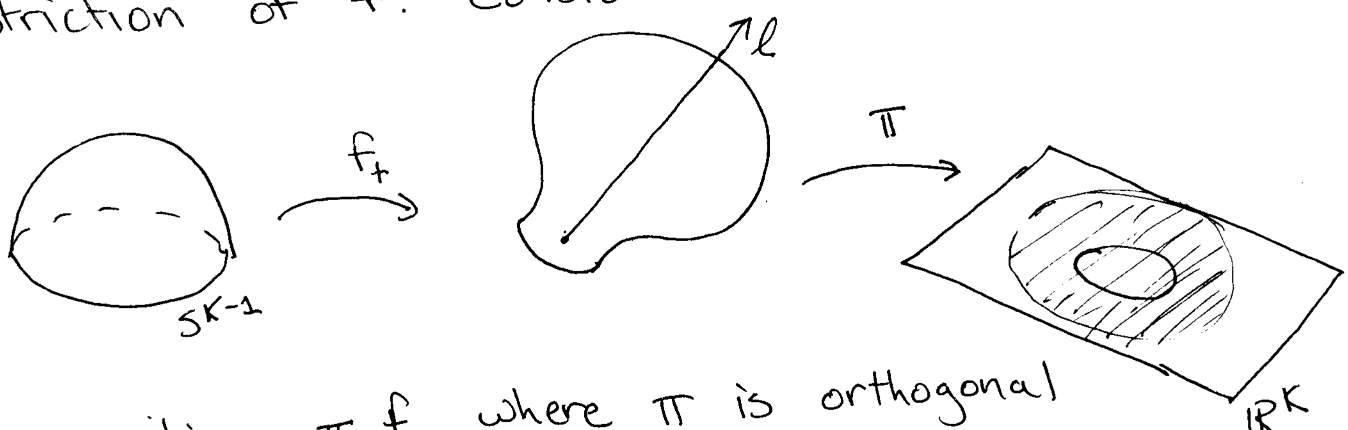
Now

$$W_2(f,0) = \deg_2\left(\frac{f}{|f|}\right) = \#\text{ pts in } \left(\frac{f}{|f|}\right)^{-1}(a) \bmod 2.$$

By symmetry,

$$\#\text{ pts in } \left(\frac{f}{|f|}\right)^{-1}(a) = \frac{1}{2}\#\text{ pts in } f^{-1}(\ell).$$

Now on the upper hemisphere, let f_+ be the restriction of f. Consider



the composition $\pi \circ f_+$ where π is orthogonal projection in the direction $\vec{a}$.

We note

(a) On S^k, $\pi \circ g(-x) = \pi(-g(x)) = -\pi(g(x))$ by symmetry of g and linearity of π.

(b) Since $\ell \cap \mathrm{Im}(g) = \emptyset$ (remember $\vec{a}$ was a regular value for $g/|g|$), we have $\pi \circ g$ avoids the origin in $\mathbb{R}^k$.

By inductive hypothesis, $\omega_2(\pi \circ g, 0) = 1$. But $\pi \circ g = \partial(\pi \circ f_+)$, so by our theorem last time,

$$\# \text{ pts in } (\pi \circ f_+)^{-1}(0) = \omega_2(\pi \circ g, 0) = 1 \mod 2$$

Thus

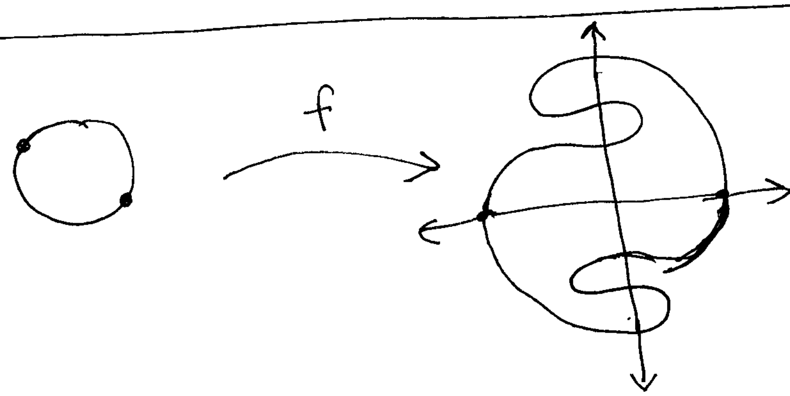
$$\begin{aligned}
1 &= \# \text{ pts in } (\pi \circ f_+)^{-1}(0) \\
&= \tfrac{1}{2} \# \text{ pts in } (\pi \circ f)^{-1}(0) && \text{(by symmetry)} \\
&= \tfrac{1}{2} \# \text{ pts in } f^{-1}(\ell) && \text{(defn of } \pi) \\
&= \# \text{ pts in } \left(\tfrac{f}{|f|}\right)^{-1}(\vec{a}) && \text{(by symmetry)} \\
&= \omega_2(f, 0). && \text{(defn of } \omega_2)
\end{aligned}$$

We can already conclude:

Theorem. If $f: S^k \to \mathbb{R}^{k+1} - \{0\}$ has $f(-x) = -f(x)$ for all x, then $\operatorname{Im} f$ intersects every line through the origin at least once.

Proof. Let ℓ be the line chosen, ~~is the~~ then $W_2(f,0) = \frac{1}{2}\,\#$ pts in $f^{-1}(\ell) = 0$. $\times\!\!\times$

Example.



Theorem. Any k smooth functions $f_1, \dots, f_k$ on S^k with $f_i(-x) = -f_i(x)$ must have a common zero.

Proof. ~~If not,~~ Consider $f: S^k \to \mathbb{R}^{k+1}$ given by

$$f(x) = (f_1(x), \dots, f_k(x), 0)$$

and let ℓ be the x_{k+1} axis.

~~Theorem~~

Consequence. Any collection of K odd homogenous polynomials in $K+1$ variables has a common real root.

We can also get the (remarkable) theorem

Theorem. For any K smooth functions on S^K there are a pair of antipodal $p, -p$ so that $g_1(p) = g_1(-p), \dots, g_K(p) = g_K(-p)$.

Proof. Let $f_i(x) = g_i(x) - g_i(-x)$. p is the common zero of the f_K.

Next class we introduce orientation!