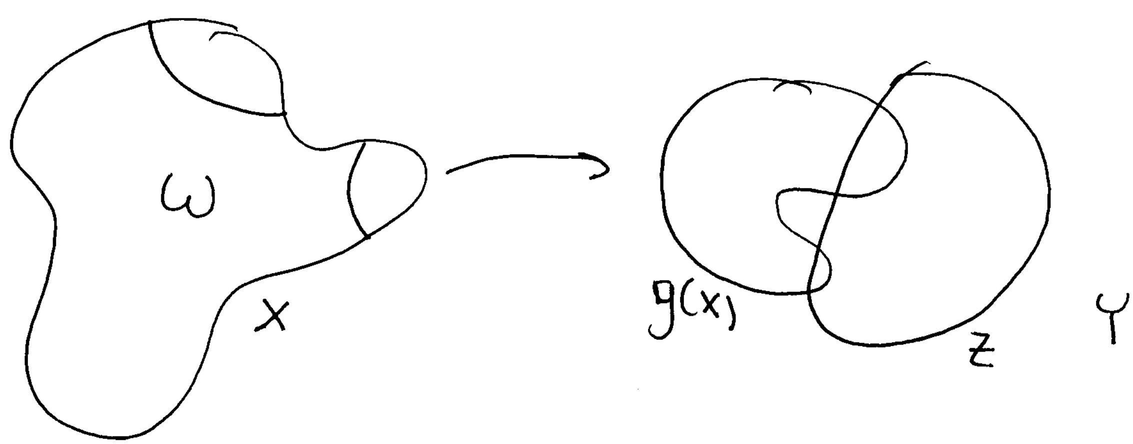
Intersection Theory mod 2

Boundary Theorem. Suppose $X = \partial W$, and $g: X \to Y$ is smooth, and $Z \subset Y$ is a closed submanifold of complementary dimension. If g extends to W, then

$$I_2(g, Z) = 0.$$

Proof.



Let $G: W \to Y$ extend g. Now consider $G^{-1}(Z)$, a 1-manifold with boundary $g^{-1}(Z)$.

We now introduce another invariant – this time for maps between manifolds of the same dimension. If X is compact and Y is connected, and $\dim X = \dim Y$, then given

$$f: X \to Y, \quad f^{-1}(\{y\}) \text{ is a 0-mfld}$$

when f is transverse to $\{y\}$.

We let

$$I_2(f, \{y\}) = \text{the mod 2 degree of } f \qquad \deg_2 f$$

(for any $\{y\}$ in Y)

Theorem. $\cancel{I_2(f,\{y\})}$ $\deg_2 f$ is well-defined.

As before, we know that $I_2(f, \{y\})$ is a homotopy invariant of f <u>for fixed y</u>. But what if we vary y?

Proof. Assume $f \pitchfork \{y_0\}$ (using transversality homotopy theorem if needed). By stack of records, $\exists$ a neighborhood U of y_0 so that

$$f^{-1}(U) = V_1 \sqcup \ldots \sqcup V_n$$

where f maps each $V_i \to U$ diffeomorphically.

For all $y \in U$, $I_2(f, \{y\}) = I_2(f, \{y_0\})$, so $I_2(f, \{y\}) = g(y)$ is a locally constant function on Y. Since Y is connected, $g(y)$ is constant, which completes proof.

We note

> If $X = \partial W$, and $f: X \to Y$ extends to W, then $\deg_2 f = 0$.

Here are some cool consequences.

Let $p: \mathbb{C} \to \mathbb{C}$ be a smooth complex function. Take some region W in $\mathbb{C}$, a 2-manifold with boundary.

We note: If $\frac{p}{|p|}: \partial W \to S^1$ has odd mod 2 degree, then p has a zero in W.

Proof. If not, then $\frac{p}{|p|}$ extends to all of W, so $\deg_z p/|p| = 0$.

We can now show

Theorem. Any complex polynomial of odd degree has a root.

Proof. ~~Such a polynomial is homotopic to~~ ~~$p(z) = z^{2k+1}$. Further, on a~~

Let $p(z) = z^{2k+1} + a_1 z^{2k} + \dots + a_{2k+1}$,

and $p_t(z) = t\,p(z) + (1-t)\,z^{2k+1}$.

Clearly, p_t is a homotopy from $p(z)$ to z^{2k+1}. For a very large ball W centered at the origin, we see that

$$\text{no } p_t(z) \text{ has a zero on } \partial W.$$

Thus $p_t/|p_t|$ provides a homotopy between $p/|p|$ and $z^{2k+1}/|z^{2k+1}|$ on ∂W, so

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$$\deg_2 \frac{p}{|p|} = \deg_2 \frac{z^{2k+1}}{|z^{2k+1}|} = 2k+1 \mod 2 = 1.$$

Thus p has a zero inside ω.

Wasn't that fun?