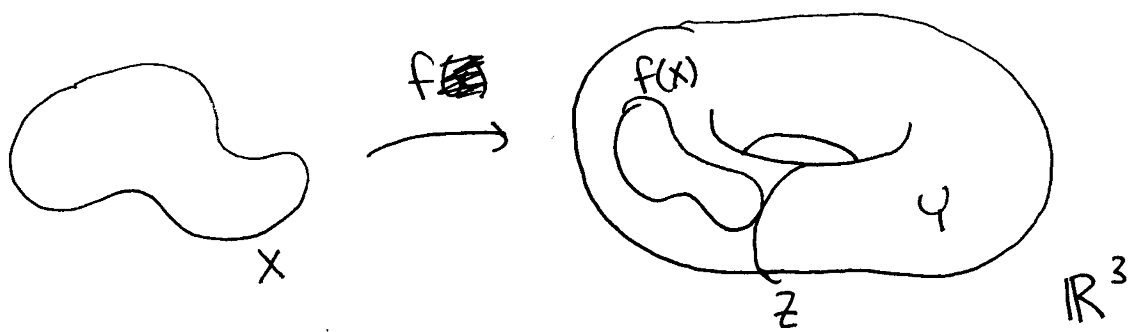
Transversality is generic – II

We now understand how to modify a map $f: X \to \mathbb{R}^n$ so that f is transversal to any given $Z \subset \mathbb{R}^n$.

What about a map $f: X \to Y$?

Idea:



Perturb $f(x)$ in $\mathbb{R}^3$, then project the image back to Y.

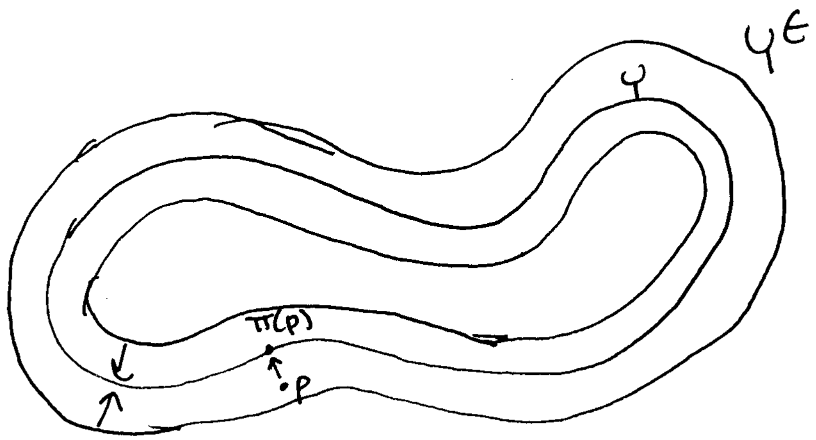
ϵ-neighborhood theorem.

For any compact $Y \subset \mathbb{R}^M$ without boundary let

$$Y^\epsilon = \text{the } \epsilon\text{-neighborhood of } Y.$$
$$= \{ p \in \mathbb{R}^M \mid d(p, Y) < \epsilon \}$$

There exists some $C > 0$ so that for all $\epsilon < C$, each $p \in Y^\epsilon$ has a unique nearest neighbor on Y, denoted $\pi(p)$.

Further, $\pi : Y^\epsilon \to Y$ is a submersion.



With the ϵ-neighborhood theorem, we can complete the proof.

Transversality Homotopy Theorem.
For any smooth map $f: X \to Y$, and any submanifold $Z \subset Y$ where Z, Y have no boundary, there exists a smooth map $g: X \to Y$ homotopic to f and with $g \pitchfork Z$, $\partial g \pitchfork$.

Proof. If $Y \subset \mathbb{R}^M$, consider the map

$$F(x,s) = \pi[f(x) + \epsilon s] : X \times \mathbb{R}^M \to Y.$$

where $\epsilon <$ the C provided by the ϵ-neighborhood theorem. We know that $F(x,s)$ is a submersion since it is a composition of submersions

$$X \times \mathbb{R}^M \xrightarrow{f + \epsilon\{s\}} \mathbb{R}^M \xrightarrow{\pi} Y$$

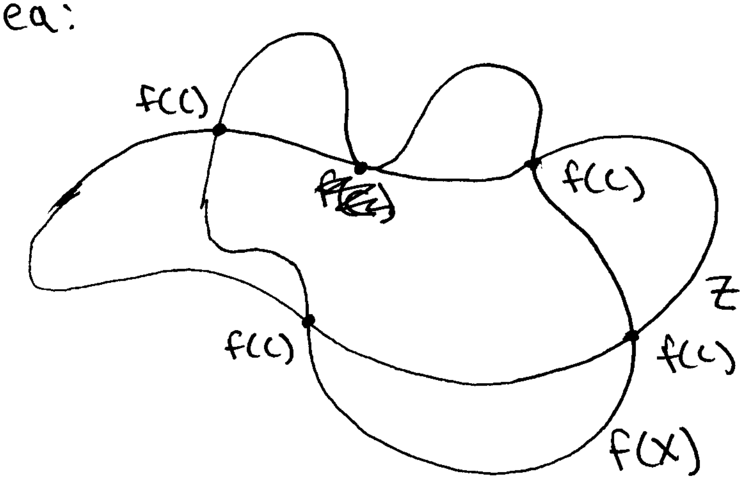
By the Transversality Theorem, the maps

$$f_s(x) = F(x,s) \quad \text{and} \quad \partial f_s(x) = \partial F(x,s)$$

are transverse to Z for almost all $s \in D^m$.
Pick such an s, and we're done.

We will want an even <u>stronger</u> theorem.

Idea:



Suppose we are transverse to Z on some $C \subset X$, but are not transverse everywhere: can we preserve the transverse part of f and vary everything else to make f transverse?

Definition. A map $f: X \to Y$ is transversal to Z <u>on C</u> $\subset X$ if

$$df_x(T_x X) + T_{f(x)} Z = T_{f(x)} Y$$

for all $x \in C \cap f^{-1}(Z)$.

Extension Theorem.

Suppose $Z \subset Y$ is a closed submanifold w/o boundary, and C is a closed subset of X. If $f: X \to Y$ has

$$f \pitchfork Z, \quad \partial f \pitchfork Z \text{ on } C \text{ (or } C \cap \partial X)$$

then $\exists$ a smooth g homotopic to f with

$$g \pitchfork Z, \quad \partial g \pitchfork Z$$

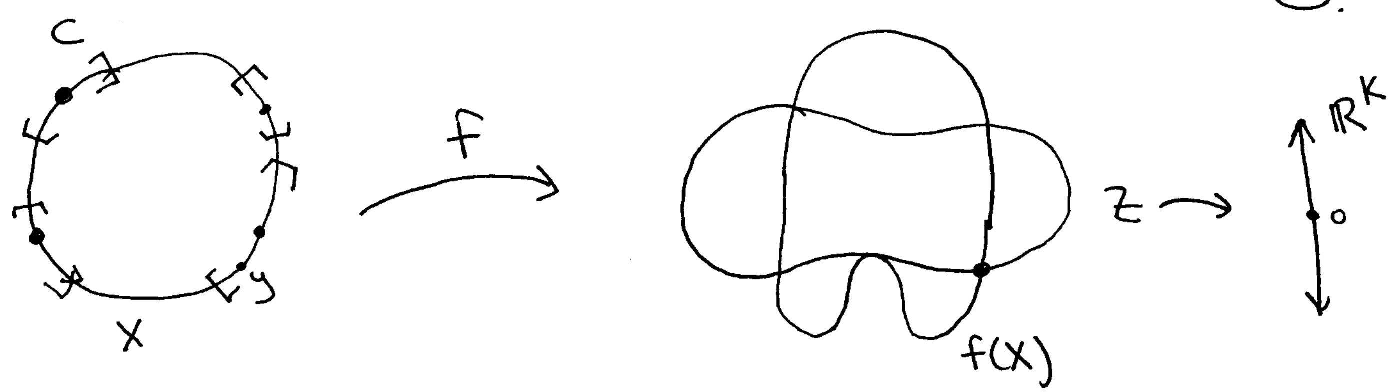
and $g = f$ on an open neighborhood of C.

To prove it, we'll need a technical lemma.

Lemma. If $U \supset C$ is an open set, $\exists$ a smooth function on X which is 1 outside U but 0 in a neighborhood of C.

Proof (of theorem).

Claim 1. $f \pitchfork Z$ in a neighborhood of C.



Pick any $y \in C$. We will show $f \pitchfork Z$ in an open neighborhood of y.

Case 1. $y \notin f^{-1}(Z)$.

Then $y \in X - f^{-1}(Z)$, which is open since $f^{-1}(Z)$ is closed.

Case 2. $y \in f^{-1}(Z)$.

As usual, we can find ~~an~~ neighborhood of $f(y)$ where Z ~~is the~~ $= g^{-1}(\{0\})$ for some map to $\mathbb{R}^K$. On this neighborhood,

$f \pitchfork Z$ at p $\iff$ $g \circ f$ has ~~0~~ p as a regular ~~value~~ point

But $g \circ f$ is regular at y, so $g \circ f$ is regular in an open neighborhood of y.

Now let γ be the function in the lemma and take $\tau = \gamma^2$. Construct

$$F : X \times D^M \longrightarrow Y$$

as before and let

$$G(x,s) = F(x, \tau(x)s).$$

We claim $G \pitchfork Z$.