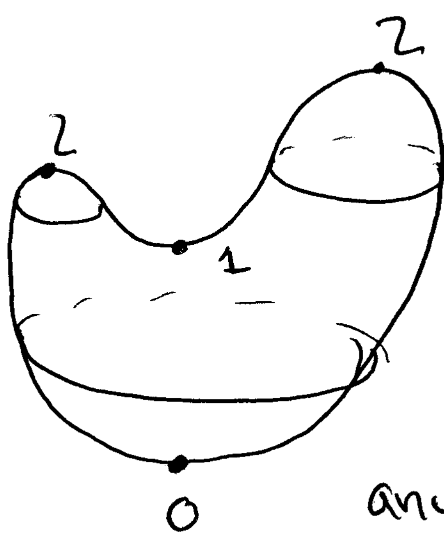
Morse Theory (completed)

We finished last class with a description of the Morse index of a critical point and a false claim about adding indices.



Here's the truth.

Theorem: Let $C(n)$ be # of critical points of any morse function $f: X \to \mathbb{R}$ of morse index n. Then

$$\sum_{i} (-1)^i C(i) = \chi(X)$$

where χ is the Euler characteristic.

In fact, we can prove this using an appealing ~~idea~~ set of ideas.

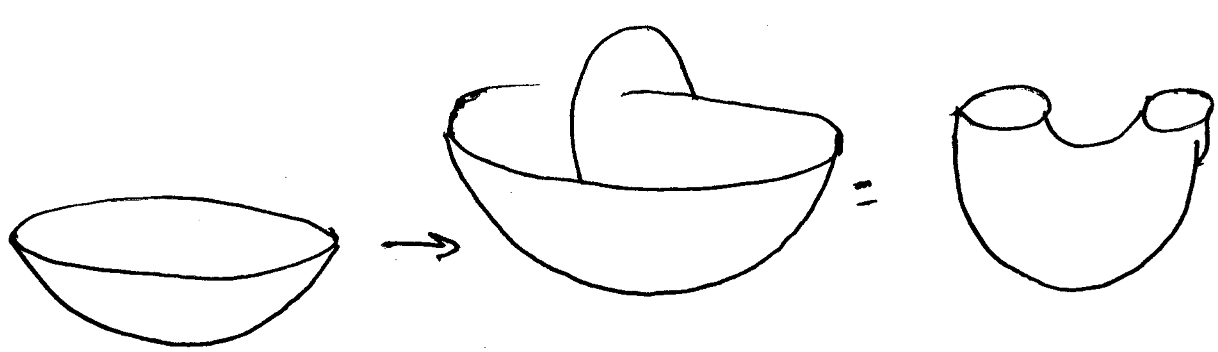
Let $X^a = f^{-1}(a)$ and $X^b = f^{-1}(b)$.

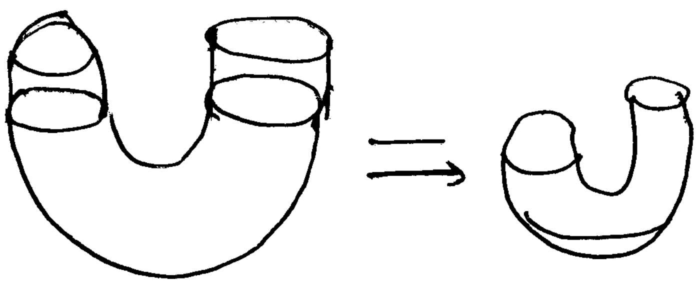
If there are no critical ~~points~~ values of f between a and b, then

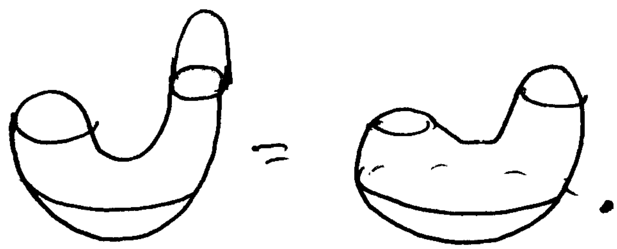
$$X^a \underset{\text{diffeo}}{\cong} X^b.$$

If there is a single critical point of index n between a and b, then

X_b is homotopic to X_a + an n-dimensional "cell"



→ 

→ 

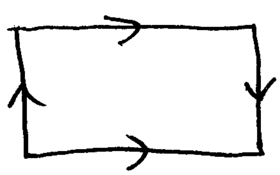
Roughly, the Euler characteristic is an alternating sum over i of the number of i-dimensional "cells" in a manifold.

For surfaces,

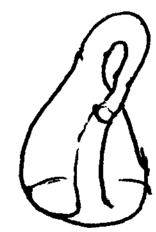
$$\chi(S) = \# \text{vertices} - \# \text{faces} + \# \text{edges}.$$

Morse theory was one clever application of Sard's theorem. Another is the Whitney embedding theorem (easy version).

We start with an n-manifold $X \subset \mathbb{R}^N$ for "some large N". How large does N need to be? Well, $N \geq n$, for sure.



Klein bottle



It can be shown (take the 8000 topology course next year!) that the Klein bottle does <u>not</u> embed in $\mathbb{R}^3$.

Whitney showed

Theorem. Any n-manifold X^n has an embedding $f : X \to \mathbb{R}^{2n}$.

This theorem is hard! But we'll prove the much easier theorem

Theorem. Any n-manifold X^n has an embedding into $\mathbb{R}^{2n+1}$.

We start with a construction.

Given $X \subset \mathbb{R}^N$, the set

$$\{(\vec{x}, \vec{v}) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \in X,\ v \in T_x X\}$$

is called the tangent bundle TX of X.

Fact: If $X \underset{f}{\cong} Y$, then $TX \simeq TY$ by the associated map (f, df).

Proposition. If X is a smooth manifold, TX is a smooth manifold with

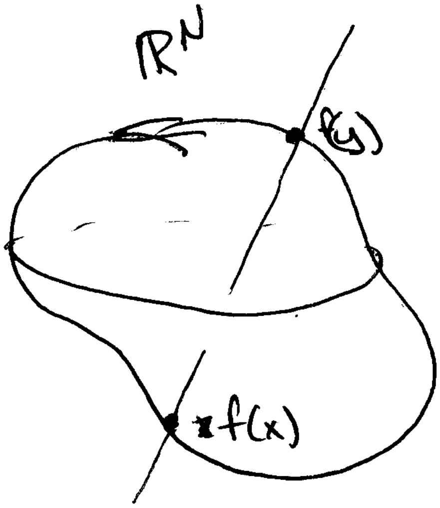
$$\dim TX = 2 \dim X.$$

Proposition. Every n-manifold X^n has an injective immersion into $\mathbb{R}^{2n+1}$.

Proof. We know that for some N, there is an injective immersion

$$f : X \to \mathbb{R}^N$$

If $N > 2k+1$, we will find some $\vec{a} \in \mathbb{R}^N$ so that if π is the projection orthogonal to $\vec{a}$, then $\pi \circ f : X \to \mathbb{R}^{N+1}$ is still an injective immersion.



Idea: The only thing that could kill injectivity is if some $f(x)$ and $f(y)$ on $f(X)$ are connected by a line in direction of a.

Let $h : X \times X \times \mathbb{R} \to \mathbb{R}^N$ be given by

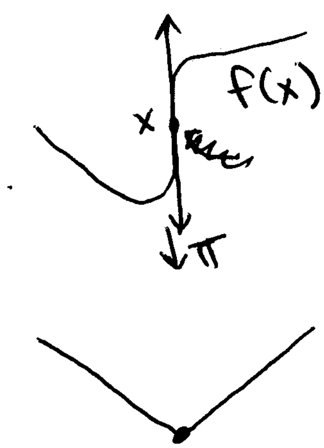
$$h(x,y,t) = t[f(x) - f(y)]$$

Then $\text{Im}\, h$ = points on lines which intersect $f(X)$ more than once. Since

$$\dim (X \times X \times I) = 2n+1 < \dim \mathbb{R}^N = N,$$

the only regular values of h are points <u>not</u> in $\text{Im}\, h$. So $\text{Im}\, h$ is a measure 0 set of $\mathbb{R}^N$, by Sard's theorem.

Idea: The only thing that could kill immersivity is if $a \in T_x X$ for some x.



So let

$$g : TX \to \mathbb{R}^N$$

be given by

$$g(x,v) = df_x(v).$$

Again, since

$$\dim TX = 2n < \dim \mathbb{R}^N = N,$$

we see $\text{Im}\, g$ is a measure 0 set of $\mathbb{R}^N$, again by Sard.