Homotopy and Stability

Having defined transverse intersections, we now want to study how they change as the manifolds X and Y move.

Definition. ~~The~~ Smooth Maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are <u>homotopic</u> if $\exists$ a smooth map

$$F_{\cancel{t}} : X \times [0,1] \to Y$$

with $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$. F is called a <u>homotopy</u>, and usually written $f_t : X \to Y$.

Lemma. Homotopy is an equivalence relation on maps from X to Y.

$$\boxed{\text{Topology is the subject where we classify things up to homotopy!}}$$

An important ~~property of~~ attribute that properties of maps can have is <u>stability</u>.

Definition A property P of maps $f: X \to Y$ is <u>stable</u> if ~~the~~ whenever f has property P, for any homotopy $f_t: X \to Y$, $\exists$ some $\epsilon > 0$ so that f_t has property P for all $t < \epsilon$. If P is stable, the class of maps with property P is a <u>stable class</u> of maps.

We will now prove the stability theorem:

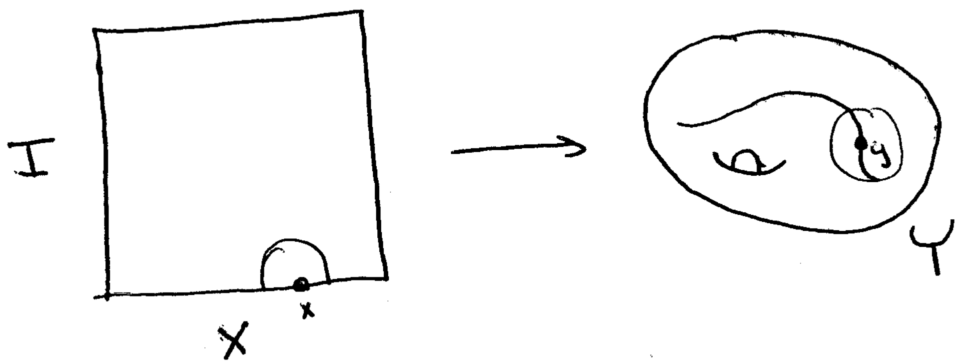
Stability Theorem. If X is compact, the following classes of maps $f: X \to Y$ are stable:

(a) local diffeomorphisms
(b) immersions
(c) submersions
(d) maps transversal to any fixed $Z \subset Y$
(e) embeddings
(f) diffeomorphisms

Proof. The proof is generally easy. We do a couple cases.

(b) immersions.

Consider the setup. $F_t : X \times I \to Y$ is a homotopy



We know $d(f_0)_x$ is injective for all x and must ~~shd~~ show $\exists\ \epsilon > 0$ so that $d(f_t)_x$ is injective for $t < \epsilon$.

Claim: It is enough to show $\exists$ an open neighborhood of each ~~$x \in X$~~ $(x,0) \in X \times I$ where $(df_t)_x$ is injective.

(True because X is compact, using pointset theorem on Lebesgue number.)

Establish local coordinates on $X \times I$ and Y around $(x,0)$ and $F(x,0) = y$.

In these coordinates,

$$(d(f_0))_x = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = \text{an } m \times n \text{ matrix with } m \geq n, \text{ and rank } n.$$

The condition that df_0 is injective is the same as the condition that

$$\det\left((d(f_0))_x^T \, d(f_0)_x \right) \neq 0.$$

But the ~~coordinate~~ entries of $d(f_t)_x$ vary continuously with x and t and det and $A \mapsto A^T A$ are continuous maps, so $\exists$ an open neighborhood of $(x,0)$ ~~ta~~ with

$$\det\left(d(f_t)_{x'}^T \, d(f_t)_{x'} \right) \neq 0,$$ as desired.

(a), (c), and (d) are similar.

(e) embeddings

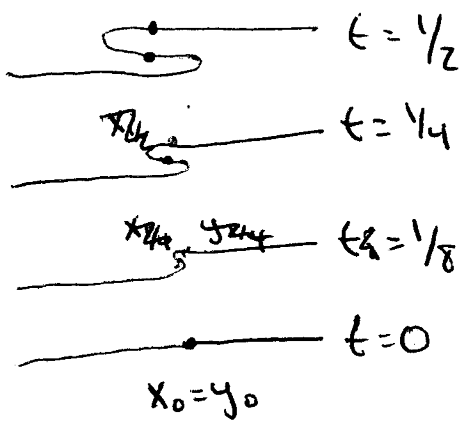
X is compact, so $f: X \to Y$ is an embedding $\Leftrightarrow$ f is a 1-1 immersion. Since immersions are stable, we need to show that given $f_t: X \to Y$, $\exists$ $\epsilon > 0$ so that f_t is 1-1.

Suppose not. $\exists$ $t_i \to 0$ and $x_i, y_i \in X$ so that $f_{t_i}(x_i) = f_{t_i}(y_i)$ (but $x_i \neq y_i$). Since X is compact, we may assume that

$$x_i \to x_0 \quad \text{and} \quad y_i \to y_0$$

But then by continuity of f_t in t,

$$f_0(x_0) = f_0(y_0) \quad \text{so} \quad x_0 = y_0.$$



Construct

$$G : X \times I \to Y \times I \quad \text{by} \quad g(x,t) = (f_t(x), t).$$

at $(x_0 = y_0, 0)$ we see (in local coordinates)

$$dG = \left(\begin{array}{c|c} df_0 & \vdots \\ \hline 0 \text{---} 0 & 1 \end{array}\right)$$

since f_0 is an immersion, df_0 has full rank so dG has full rank. Thus G is a local immersion near $(x_0, 0)$. But this contradicts our ~~[illegible]~~ previous construction of

$$(x_i, t_i), (y_i, t_i) \to (x_0, 0)$$

with $G(x_i, t_i) = G(y_i, t_i)$.