Review of linear functions and functionals.

Recall that a map $f: V \to \mathbb{R}$ from a vector space to $\mathbb{R}$ is called a <u>linear functional</u> if it is a linear map.

If V is finite-dimensional, then the space of linear functionals on V, called the <u>dual space</u> V^* is isomorphic to V.

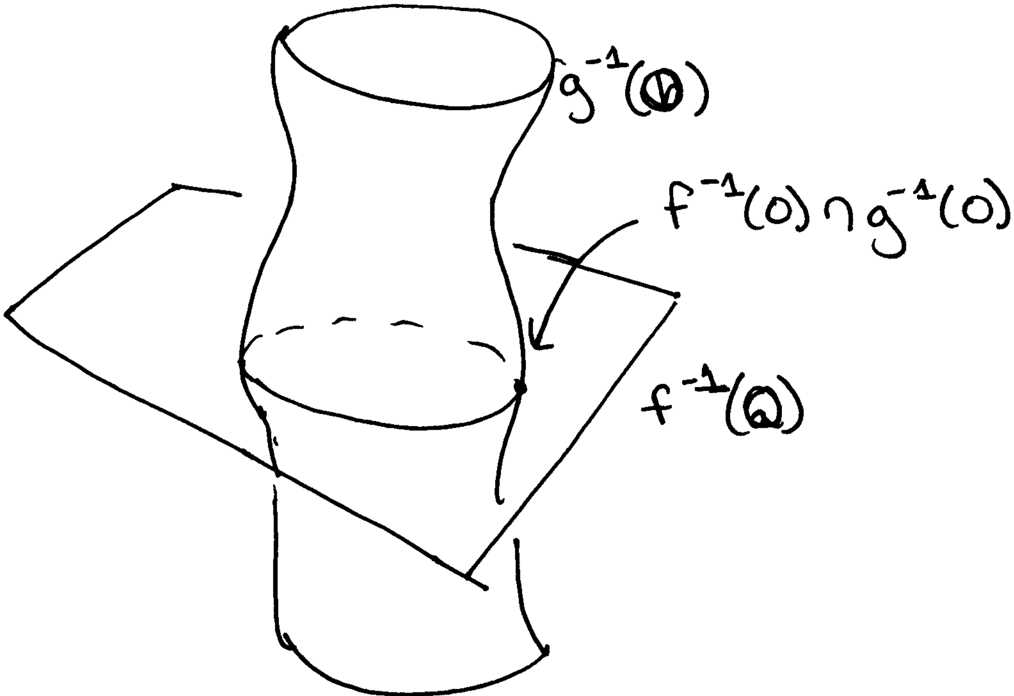
Choose a basis $e_1, \dots, e_n$ for V and a corresponding inner product s.t. the e_i are orthonormal. Then each functional

$$f(\vec{v}) = \langle \vec{v}, \vec{f} \rangle \quad \text{where } \vec{f} = (f(e_1), \dots, f(e_n)).$$

We say $f_1, \dots, f_k$ are linearly independent $\Leftrightarrow$ $\vec{f}_1, \dots, \vec{f}_k$ are linearly independent.

Given a collection of k vectors in $\mathbb{R}^n$, $v_1, \dots, v_k$, they are lin. indep. $\Leftrightarrow$ the Gram determinant

$$\det \begin{pmatrix} \langle v_1, v_1 \rangle & \dots & \langle v_1, v_k \rangle \\ \\ \langle v_k, v_1 \rangle & \dots & \langle v_k, v_k \rangle \end{pmatrix} \neq 0.$$

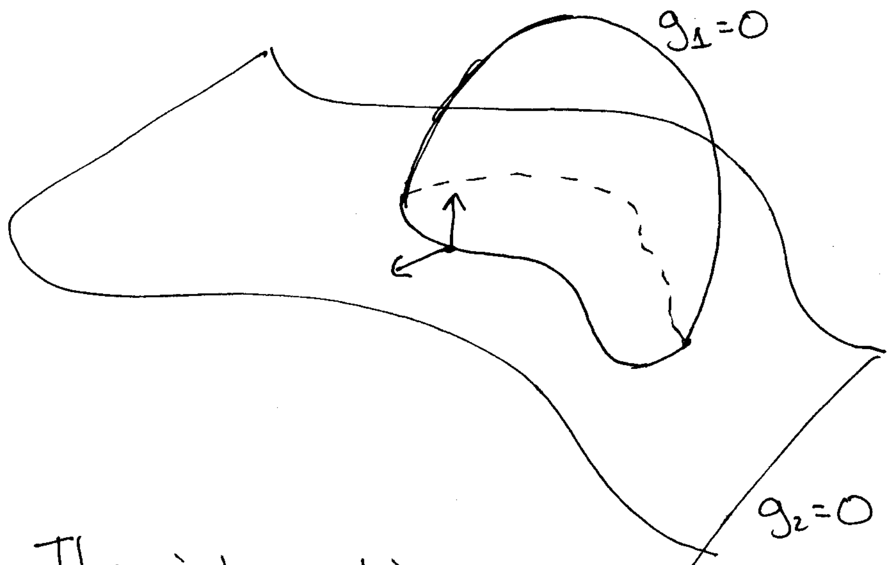


Given smooth functions $g_1, \dots, g_\ell$ on X, when does the set Z of x s.t. $g_i(x) = 0$ for all i look nice?

Definition. A set of smooth functions $g_1, \dots, g_\ell$ on X is <u>independent</u> at x if the linear functionals $d(g_1)_x, \dots, d(g_\ell)_x$ are linearly independent.

Proposition. If the smooth functions $g_1, \dots, g_\ell$ on X are independent where they vanish, the set of common zeros of the g_i is a submanifold of X of dimension $\dim X - \ell$.

The picture here is easy to understand:



The intersection of $g_1=0$ and $g_2=0$ is a manifold $\Leftrightarrow$ ∇g_1 and ∇g_2 are linearly independent everywhere on the intersection.

Note: We speak of such a set as having "codimension" ℓ.

(See algebraic geometry...)

Example. The set of equilateral triangles of side 1 is a 3-dimensional submanifold of the ~~space~~ 6-dimensional manifold of triangles in $\mathbb{R}^2$.

We have

$$X = \text{triangles in } \mathbb{R}^2 \underset{\text{diffeo}}{\cong} \mathbb{R}^6$$

Let $g_1(x,y,z) = |x-y|^2 - 1$. $g_3 = |z-x|^2 - 1$

$g_2(x,y,z) = |y-z|^2 - 1$.

To show: g_1, ~~and~~ g_2, g_3 are independent when they are all zero.

Step 1. Compute the differentials.

$$dg_1 = [\, 2(x_1-y_1) \quad 2(x_2-y_2) \quad -2(x_1-y_1) \quad -2(x_2-y_2) \quad 0 \quad 0 \,]$$

$$dg_2 = [\, 0 \quad 0 \quad 2(y_1-z_1) \quad 2(y_2-z_2) \quad -2(y_1-z_1) \quad -2 \,]$$

$$dg_3 = [\, -2(z_1-x_1) \quad -2(z_2-x_2) \quad 0 \quad 0 \quad 2(z_1-x_1) \quad 2(z_2-x_2) \,]$$

Step 2. Prove they are linearly independent.

We write $a_i = x_i - y_i$, $b_i = y_i - z_i$, $c_i = z_i - x_i$ and eliminate 2's to see we must consider

$$\tfrac{1}{2}\,dg_1 = [a_1 \;\; a_2 \;\; -a_1 \;\; -a_2 \;\; 0 \;\; 0]_x$$

$$\tfrac{1}{2}\,dg_2 = [0 \;\; 0 \;\; b_1 \;\; b_2 \;\; -b_1 \;\; -b_2]_y$$

$$\tfrac{1}{2}\,dg_3 = [-c_1 \;\; -c_2 \;\; 0 \;\; 0 \;\; c_1 \;\; c_2]_z$$

Computing the Gram matrix

$\cancel{x a_1 - z c_1 = 0.}$
$\cancel{x a_2 - z c_2 = 0.}$
$\cancel{-x c_1 + y b_1 = 0}$

$$\begin{bmatrix} 2a_1^2 + 2a_2^2 & -a_1 b_1 - a_2 b_2 & -a_1 c_1 - a_2 c_2 \\ & 2b_1^2 + 2b_2^2 & -b_1 c_1 - b_2 c_2 \\ & & 2c_1^2 + 2c_2^2 \end{bmatrix}$$

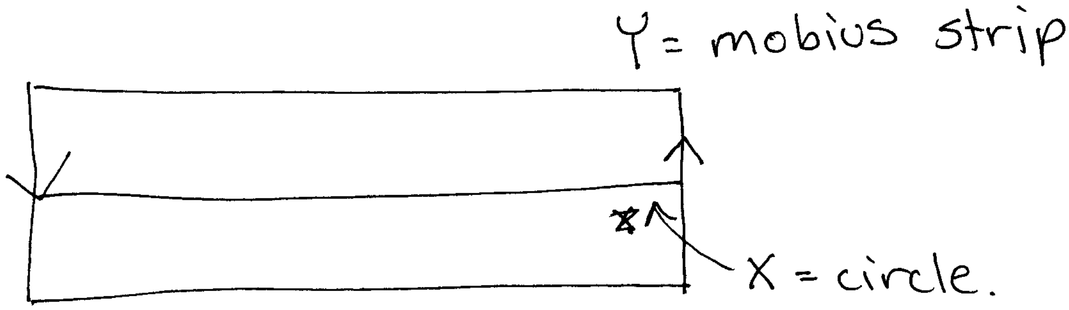
So the determinant is, using Maple,

$$2(a_1^2 + a_2^2)\left[4(b_1^2+b_2^2)(c_1^2+c_2^2) - (b_1c_1+b_2c_2)^2\right]$$

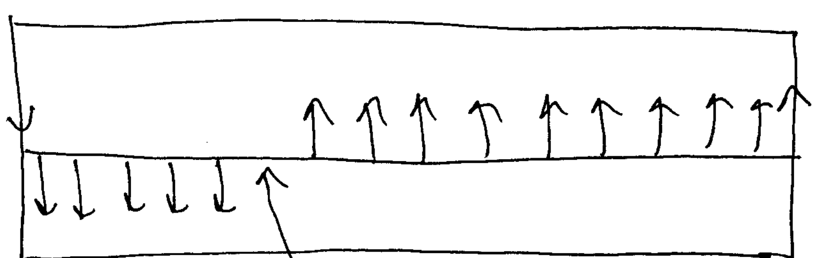
$$(a_1b_1 + a_2b_2)\left[(-a_1b_1 - a_2b_2)(2c_1^2 + 2c_2^2) - (a_1c_1 + a_2c_2)(b_1c_1+b_2c_2)\right]$$

$$+ (-a_1c_1 - a_2c_2)\left[(a_1b_1 + a_2b_2)(b_1c_1+b_2c_2) - (-a_1c_1 - a_2c_2)(2b_1^2 + 2b_2^2)\right]$$

Is it true that <u>every</u> submanifold $X \subset Y$ is the preimage of a regular value of some map $f: Y \to Z$? No! ~~But it's not trivial to see why~~



If $X = f^{-1}(y)$, then $f: Y \to \mathbb{R}$ by dimension counting and ∇f is $\perp$ to X everywhere. Further, ∇f is nonzero on X since y is a regular value. So let's draw ∇f...

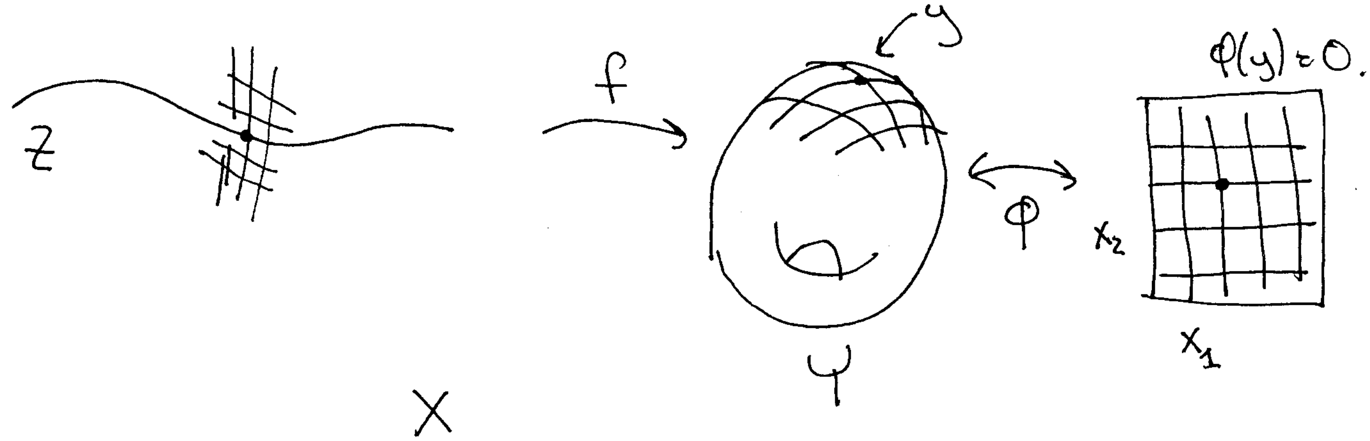


Yikes! ∇f points <u>both</u> ways? X

Similarly, it is not true that every $Z \subset X$ is cut out by independent functions on X. But we <u>do</u> have

Proposition. If y is a regular value of a smooth map $f: X \to Y$ then $f^{-1}(y)$ is the ~~com~~ set of common zeros of a collection of independent functions on X.

Proof.



Choose local coordinates around y and take the coordinate functions $h_1, \dots, h_\ell$ composed with f to get

$$g_1 = h_1 \circ f, \quad \dots, \quad g_\ell = h_\ell \circ f.$$