4

5 - Submersions.

We have just dealt with the case

$f: X \to Y$ where $\dim Y > \dim X$.

What if $\dim Y < \dim X$?

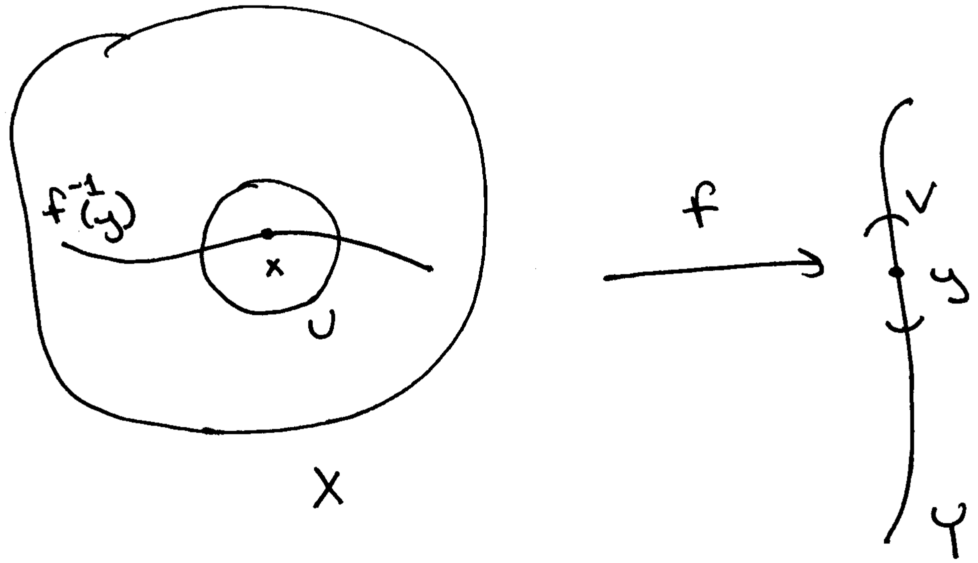
Definition. If df_x is surjective, then f is a <u>submersion</u> at x.

As before, we can choose local coordinates to provide a canonical form for such a map.

Local Submersion Theorem. If $f: X \to Y$ is a submersion at x and $y = f(x)$ then $\exists$ local coordinates so that $f(x_1, \dots, x_k) = (x_1, \dots, x_l)$.

Proof. Much like that of local immersion theorem.

Now recall that $\{x \in X \mid f(x) = y\}$ is called the preimage of y. In general, $f^{-1}(y)$ can be pretty weird — but not for submersions!



If f is a submersion at x, choose the local coordinates which make $f(x_1, \ldots, x_k) = (x_1, \ldots, x_\ell)$. Then $f^{-1}(0)$ is the set of points $(0, \ldots, 0, x_{\ell+1}, \ldots, x_k)$, which is diffeomorphic to $\mathbb{R}^{k-\ell}$.

Definition. A point $y \in Y$ is a <u>regular value</u> for $f: X \to Y$ if df_x is a submersion at all x in $f^{-1}(y)$.

~~This proves~~
We then have

Preimage Theorem. If y is a regular value of $f: X \to Y$ then $f^{-1}(y)$ is a submanifold of X of dimension $\dim X - \dim Y$.

If y is not a regular value it is a <u>critical value</u>.

(If $y \notin \operatorname{Im} f$, then y <u>is</u> a regular value, but $f^{-1}(y) = \emptyset$, so we have to claim that "$\dim \emptyset = n$" is a true statement for any n).

Examples. S^n is an n-dimensional manifold.

Consider $f: \mathbb{R}^{n+1} \to \mathbb{R}$ given by
$f(x_1, \dots, x_{n+1}) = x_1^2 + \dots + x_{n+1}^2$.
We compute

$$df = (2x_1 \cdots 2x_{n+1})$$

which is the 0 vector $\Leftrightarrow$ all $x_i = 0$. So for any nonzero y,

$$f^{-1}(y) = \text{sphere of radius } \sqrt{y}$$
$$= \text{a submanifold of } \mathbb{R}^{n+1} \text{ of dimension } \dim(\mathbb{R}^{n+1}) - \dim(\mathbb{R}) = n.$$

Example. The orthogonal group $O(n)$ is a manifold.

Recall $A \in O(n)$ if $AA^T = I$.

Step 1. $O(n) \subset M(n)$, where $M(n)$ is the space of $n \times n$ matrices. $M(n) \underset{\text{diffeo}}{\cong} \mathbb{R}^{n^2}$. $\begin{pmatrix} \cdot & \cdots & \cdot \\ \vdots & & \vdots \\ \cdot & \cdots & \cdot \end{pmatrix}$

Step 2. Let $S(n)$ be the space of <u>symmetric</u> $n \times n$ matrices. $S(n) \underset{\text{diffeo}}{\cong} \mathbb{R}^{n(n+1)/2}$. $\begin{pmatrix} \cdot & \cdots & \cdot \\ & \ddots & \vdots \\ & & \cdot \end{pmatrix}^{n}_{1}$

Step 3. Let $f : M(n) \to S(n)$ be given by

$$f(A) = AA^T.$$

We see f is smooth, $O(n) = f^{-1}(I)$.

Step 4. We must show that I is a regular value of f. We compute

$$df_A(B) = \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s}$$

$$= \lim_{s \to 0} \frac{(A+sB)(A+sB)^T - AA^T}{s}$$

$$= \lim_{s \to 0} \frac{(A+sB)(A^T+sB^T) - AA^T}{s}$$

$$= BA^T + AB^T$$

Is df surjective when $f(A) = I$? Well,

$$T_A M(n) = M(n)$$

and (since $S(n)$ is a linear subspace of $M(n)$)

$$T_{f(A)} S(n) = S(n).$$

So the question is: If $f(A) = I$, and $C \in S(n)$, does there exist $B \in M(n)$ so that

$$BA^T + AB^T = C \ ?$$

We can rewrite this as

$$BA^T + (BA^T)^T = C = \tfrac{1}{2}C + \tfrac{1}{2}C^T. \quad (\text{since } C \in S(n))$$

Solving for B in

$$BA^T = \tfrac{1}{2}C$$

$$BA^TA = \tfrac{1}{2}CA$$

$$B = \tfrac{1}{2}CA$$

We come up with the "guess" $B = \frac{1}{2} CA$.
We then check

$$BA^T + AB^T = \left(\tfrac{1}{2} CA\right) A^T + A\left(\tfrac{1}{2} CA\right)^T$$

$$= \tfrac{1}{2} C\,(AA^T) + \tfrac{1}{2} A(A^T C^T)$$

$$= \tfrac{1}{2} C + \tfrac{1}{2} C^T = C.$$

- Step 5. Thus $O(n)$ is a submanifold of dimension

$$\dim O(n) = \dim M(n) - \dim S(n)$$
$$= n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Remark. $O(n)$ is also a group, of course. And

$m: O(n) \times O(n) \to O(n)$, $m(A,B) = AB$
$i: O(n) \to O(n)$, $i(A) = A^{-1}$

are smooth maps. A manifold which is a group and has smooth operations is called a Lie group.