3 - Facts from vector calculus.

We now ~~3~~ review some helpful facts from vector calculus.

Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map

$$df_x(\vec{h}) = \text{directional derivative in direction } \vec{h}$$

$$= \lim_{t \to 0} \frac{f(\vec{x} + t\vec{h}) - f(\vec{x})}{t}$$

This limit is linear in h (why?) and we can write it as a matrix-vector product

$$(df_x)\, h = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

where the matrix of first partials is also known as the Jacobian matrix. This generalizes the ordinary (scalar) directional derivative for $f: \mathbb{R}^n \to \mathbb{R}^1$.

$$df(h) = \left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \nabla f \cdot \vec{h}.$$

We prefer to think

$$df_x : \mathbb{R}^n \to \mathbb{R}^m \quad \text{is a linear map.}$$

Chain Rule. Given

$$\begin{tikzcd} U \subset \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g \circ f = \text{``}g(f)\text{''}"'] & V \subset \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^\ell \end{tikzcd}$$

We have

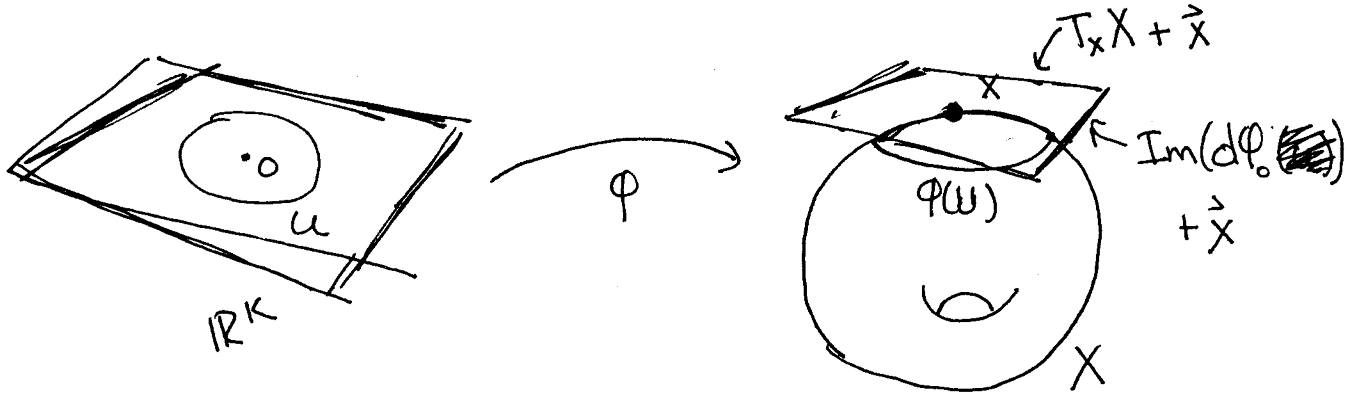
$$d(g \circ f)_x = dg_{f(x)} \circ df_x \quad \leftarrow \text{composition of linear maps}$$

$$= \left(dg_{f(x)}\right)\left(df_x\right) \quad \leftarrow \text{matrix product}$$

Even in multivariable calculus, the derivative is the "best linear approximation" of a map in the sense that

$$\lim_{t \to 0} \frac{f(\vec{x} + t\vec{h}) - f(\vec{x}) - tf'(x)}{t^2} = 0.$$

So suppose we have a local parametrization of a smooth ~~man~~ K-dimensional manifold X.



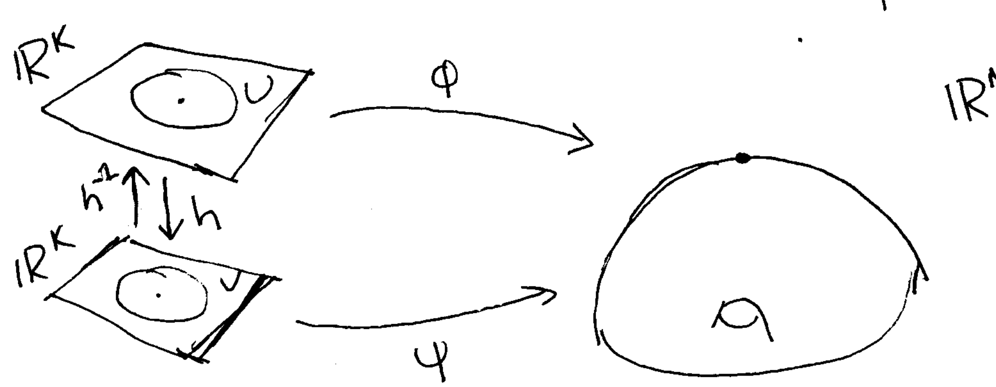
we can approximate $\varphi(u)$ by $\underset{\vec{x}}{\varphi(0)} + d\varphi_0(u)$, which is a map whose image is a translate of the subspace $\mathrm{Im}(d\varphi_0)$. We define

$$T_x(X) \text{ or } T_xX = \mathrm{Im}(d\varphi_0)$$

to be the <u>tangent space</u> to X at x.

Proposition. The tangent space does not depend on the choice of parametrization.

Proof. Suppose we have two parametrizations



~~Consider~~

Let h denote $\psi^{-1} \circ \varphi$. Then we have $h^{-1} = \varphi^{-1} \circ \psi$. Now

$$\psi \circ h = \psi \circ (\psi^{-1} \circ \varphi) = \varphi$$

so

$$d(\psi \circ h) = d\varphi$$

But by the chain rule

$$d(\psi \circ h) = d\psi \circ dh.$$

So since

$$d\varphi = d\psi \circ dh, \quad \mathrm{Im}(d\varphi) \subset \mathrm{Im}(d\psi).$$

But similarly $\mathrm{Im}(d\psi) \subset \mathrm{Im}(d\varphi)$. So

$$\mathrm{Im}(d\varphi) = \mathrm{Im}(d\psi)$$ and we're done.

It is not hard to show that

$$\dim T_x M = \dim M.$$

Now suppose we have a map

$$f : X \longrightarrow Y$$

where X and Y are smooth manifolds.
We want to define a ~~map~~ linear map

$$df_x : T_x X \longrightarrow T_{f(x)} Y$$

so that the chain rule holds. Suppose $\varphi : U \to X$ and $\psi : V \to Y$ are local parametrizations of X and Y. Then we can write

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, "\varphi"] \arrow[r, "h = \psi^{-1} \circ f \circ \varphi"'] & V \arrow[u, "\psi"']
\end{tikzcd}$$

This is called a "commutative diagram" because the maps defined by following different combinations of arrows are the same. (mention <u>diagram</u> <u>chase</u>).

We get a similar diagram of <u>derivatives</u>:

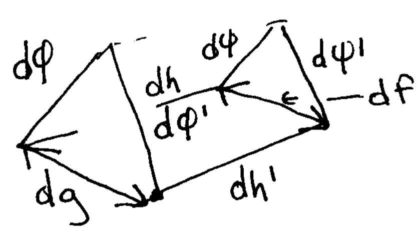
$$\begin{tikzcd} T_xX \arrow[r, dashed, "df"] & T_yY \\ \mathbb{R}^k \arrow[u, "d\varphi_0"] \arrow[r, "dh_0"] & \mathbb{R}^\ell \arrow[u, "d\psi_0"'] \end{tikzcd}$$

This leads us to <u>define</u>

$$df = d\psi_0 \circ dh_0 \circ d\varphi_0^{-1}.$$

This definition is <u>required</u> in order to make the diagram commute, since $\mathrm{Im}\, d\varphi_0 = T_xX$ (if $\mathrm{Im}\, d\varphi_0 \subsetneq T_xX$, some of the mapping df could be changed w/o messing up commutativity).

Thought exercise: How do we know this doesn't depend on our choice of parametrization?



This is really a theorem about comm. diagrams!

What about the chain rule? Well, given

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\ U \arrow[u, "\varphi"] \arrow[r, "h"'] & V \arrow[u, "\psi"] \arrow[r, "j"'] & W \arrow[u, "\eta"'] \end{tikzcd}$$

we can build the square

$$\begin{tikzcd} X \arrow[r, "g \circ f"'] & Z \\ U \arrow[u, "\varphi"] \arrow[r, "j \circ h"] & W \arrow[u, "\eta"'] \end{tikzcd}$$

Now according to our definition

$$d(g \circ f)_x = d\eta_0 \circ d(j \circ h)_0 \circ d\varphi_0^{-1}$$

But using the ordinary chain rule

$$d(j \circ h)_0 = dj_0 \circ dh_0 ,$$

so

$$d(g \circ f)_x = \underbrace{d\eta_0 \circ dj_0}(\circ d\psi_0^{-1} \circ \underbrace{d\psi_0) \circ dh_0 \circ d\varphi_0^{-1}}$$

$$= dg_{f(x)} \circ df_x$$