Differential Topology - Foundations.

We start with some fundamental definitions:

Definition. A map $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ (open subset) is <u>smooth</u> if it is C^∞ (has continuous partials of all orders).

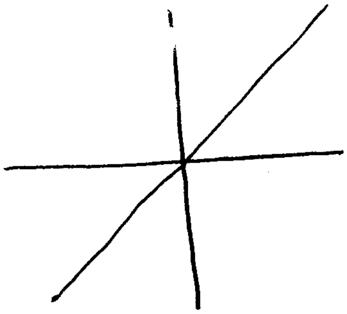
(We rarely use more than 1 derivative.)

A map $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ (any subset) is smooth if $\exists$ a smooth map F on an open set $U \supset X$ so that $F : U \to \mathbb{R}^m$ and $F = f$ on X.

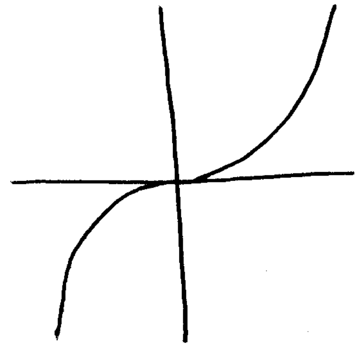
The most important smooth maps will be <u>diffeomorphisms</u>:

Definition. If $f : X \to Y$ and $f^{-1} : Y \to X$ are smooth bijections, then f is a diffeomorphism.

Warning! A smooth homeomorphism is <u>not</u> always a diffeomorphism.



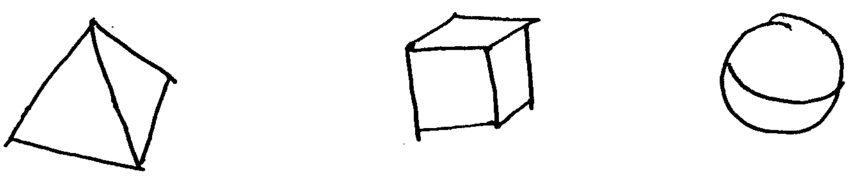
x



x^3

Smoothness is an example of a concept depending only on the behavior of something in an open neighborhood of a point. We call these things <u>local</u>.

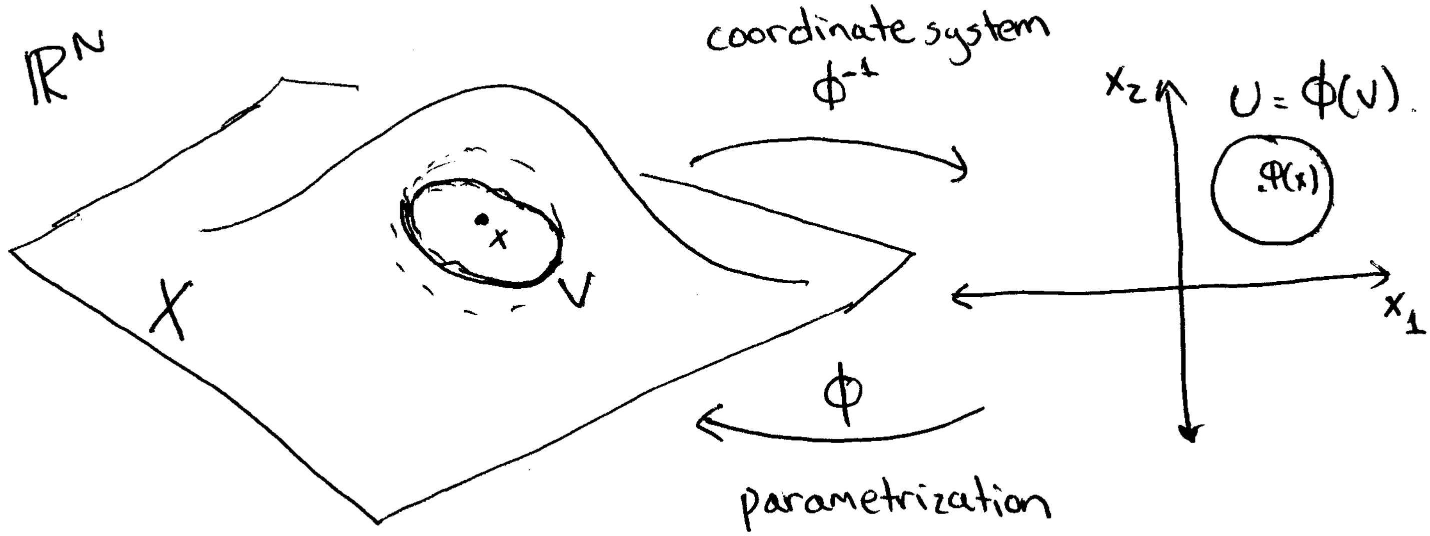
(As opposed to <u>global</u>.)



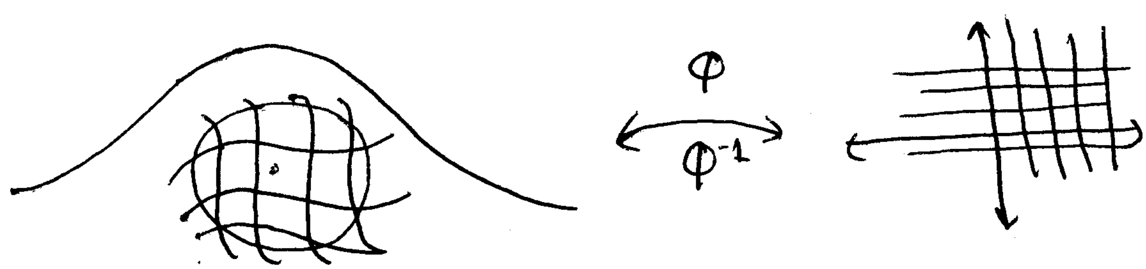
Which are diffeomorphic?

We now define manifolds as subspaces of some Euclidean $\mathbb{R}^N$. (We could also prove all the results of this course for abstract manifolds.)

Definition. Let $X \subset \mathbb{R}^N$ be any subset. If X is locally diffeomorphic to $\mathbb{R}^k$ then X is a smooth k-dimensional manifold.

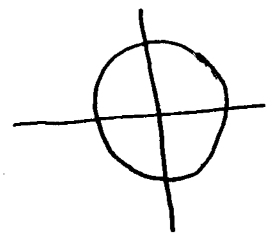


We often think of the maps ϕ, ϕ^{-1} as



a way to refer to points in X by <u>local coordinates</u> $(x_1, \ldots, x_k)$.

(But we're actually a little sloppy - do you see how?)

Example.  The unit circle in $\mathbb{R}^2$ is a smooth manifold.

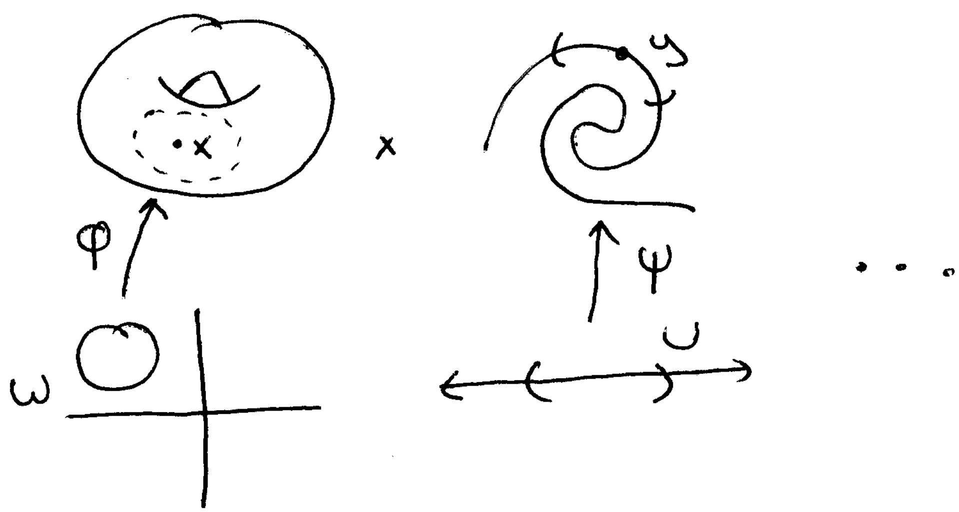
(Use $\arctan(y,x)$ and coordinates in two neighborhoods.)

~~Definition. The product of two topological spaces X, Y is the set of ordered pairs (x,y) with topology generated by . . .~~

Recall that if $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ we can form $X \times Y \subset \mathbb{R}^N \times \mathbb{R}^M = \mathbb{R}^{N+M}$.

Proposition. If X, Y are smooth manifolds then $X \times Y$ is a smooth manifold and $\dim X \times Y = \dim X + \dim Y$.

Proof. Suppose $\dim X = k$, $\dim Y = \ell$.



Last, we have

Definition. If $Z \subset X \subset \mathbb{R}^n$ and Z, X are manifolds, then Z is a submanifold of X.

HwK 3, 5, 18, p. 7.