Relationships between Means

We let Min a and Max a be the minimum and maximum values of (a).

Proposition. Min $a < M_r(a) <$ Max a, unless all the a are equal, or $r<0$ and some $a=0$. ~~Ex which~~

If all a are equal, Min $a = M_r(a) =$ Max a, and if $r<0$ and some $a=0$, Min $a = 0 = M_r(a)$ and $M_r(a) \leq$ Max a.

Proof. Suppose $\sum p = 1$, so we write the weights as q. Since

$$U(a) = \frac{\sum qa}{\sum q} = \sum qa,$$

we have

$$\begin{aligned} 0 = \sum qa - U(a) &= \sum qa - \left(\sum q\right)U \\ &= \sum qa - \sum qU \\ &= \sum q(a-U). \end{aligned}$$

This equation means that either

1) all the a_i are = to U

or

2) some $a_i > U$ and some $a_j < U$

This proves the theorem if $r=1$.

If $a>0$ or $r>0$, we note

$$\left(M_r(a)\right)^r = U(a^r)$$

but then

$$\text{Min } a^r < \left(M_r(a)\right)^r < \text{Max } a^r$$

so if $r>0$, $f(x)=x^r$ is increasing and

$$(\text{Min } a)^r < \left(M_r(a)\right)^r < (\text{Max } a)^r$$

and if $r<0$, $f(x)=x^r$ is decreasing and

$$(\text{Max } a)^r < \left(M_r(a)\right)^r < (\text{Min } a)^r$$

In the first case, taking rth roots preserves the sense of the inequalities, so

$$\text{Min } a < M_r(a) < \text{Max } a$$

and in the second taking rth roots reverses the sense of the inequality (again, $g(x) = x^{1/r}$ is decreasing if $r < 0$), so

$$\text{Min } a < M_r(a) < \text{Max } a.$$

$$\therefore$$

We now understand the M_r as different "averages" of the (a) which obey a fundamental property. We see that $G(a)$ behaves similarly next.

Proposition. $\text{Min } a < G(a) < \text{Max } a$, unless all the a are equal or some a is zero.

Proof. If some $a = 0$, then $G = 0$, and $\text{Min } a = 0 = G(a) \leq \text{Max } a$. Otherwise, $G > 0$. In that case, again taking weights q s.t. $\sum q = 1$,

$$G = \prod a_i^{q_i}$$

~~so~~ but

$$G = G^{\sum q_i} = \prod G_i^{q_i}$$

so our first equation $\Rightarrow$

$$1 = \frac{\prod a_i^{q_i}}{\prod G^{q_i}} = \prod \left(\frac{a_i}{G}\right)^{q_i}$$

Thus either

1) every $a_i = G$

or

2) some $\frac{a_i}{G} > 1$ and some $\frac{a_j}{G} < 1$.

So we're done.

We now see that G is the "missing element" of the M_r family— really it would be fair to call G by the name M_0.

Proposition. $\lim_{r \to 0} M_r(a) = G(a)$.

Proof. Observe that if every $a > 0$,

$$M_r(a) = \left(\sum q a^r \right)^{1/r}$$

$$= \exp\left(\log\left(\left(\sum q a^r\right)^{1/r}\right)\right)$$

$$= \exp\left(\frac{1}{r} \log \sum q a^r\right).$$

Now consider the Taylor expansion of

$$f(x) = q a^x, \quad \text{around } x = 0$$

We have

$$f'(x) = q a^x \log a, \quad f'(0) = q \log a$$

so

$$f(x) = q + (q \log a) x + O(x^2)$$

So

$$\sum q a^r = \sum \left(q + q \log a \, r + O(r^2)\right)$$

$$= 1 + r \sum q \log a + O(r^2)$$

and

$$\exp\left(\frac{1}{r} \log \sum q a^r\right) = \exp\left(\frac{1}{r} \log\left(1 + r \sum q \log a + O(r^2)\right)\right).$$

Now as $r \to 0$, we have

$$\lim_{r \to 0} \frac{\log(1 + r \sum q \log a + O(r^2))}{r}$$

which is equal (by L'Hospital's rule) to

$$\lim_{r \to 0} \frac{\sum q \log a + O(r)}{1 + r \sum q \log a + O(r^2)} = \sum q \log a .$$

So we have

$$\lim_{r \to 0} M_r(a) = \exp\left(\sum q_i \log a\right)$$

$$= \prod a^q = G(a). \quad \therefore$$

We now show that Min a and Max a are also "members of the family". To see why, consider

Example. $a = (1, 6)$, $q = (1/2, 1/2)$

$$M_r(a) = \left(\frac{1^r + 6^r}{2}\right)^{1/r} = \left(\frac{1}{2} + \frac{6^r}{2}\right)^{1/r}$$

Now as $r \to \infty$, we see that $\frac{6^r}{2} >> \frac{1}{2}$,

so

$$\lim_{r \to \infty} M_r(a) = \lim_{r \to \infty} \left(\frac{6^r}{2}\right)^{1/r} = \lim_{r \to \infty} \frac{6}{2^{1/r}} = 6.$$

That is, a larger number in a becomes <u>much</u> larger as we increase r, and eventually dominates the entire calculation.

Proposition. $\lim_{r\to\infty} M_r(a) = \operatorname{Max} a,$

$$\lim_{r\to -\infty} M_r(a) = \operatorname{Min} a.$$

Proof. If a_k is (one of) the largest a, then we know $M_r(a) \le a_k$. Further, we know that

$$M_r(a) = \left(\sum q_i a_i^r\right)^{1/r}$$

$$\ge \left(q_k a_k^r\right)^{1/r}$$

$$\ge q_k^{1/r} a_k.$$

But as $r\to\infty$, $1/r \to 0$, so $q_k^{1/r} \to q_k^0 = 1$. This proves

$$\lim_{r\to\infty} M_r(a) = \operatorname{Max}(a).$$

If any a is zero, $M_r(a) = 0$ for $r<0$, so the second equation is true. If not,

$$M_{-r}(a) = \frac{1}{M_r(1/a)}$$

so if a_k is one of the smallest (a) then we have $1/a_k$ is one of the largest $1/a$.

Now then

$$q_k^{1/r}(1/a_k) \leq M_r(1/a) \leq 1/a_k$$

or

$$q_k^{1/r}(1/a_k) \leq \frac{1}{M_{-r}(a)} \leq 1/a_k$$

or

$$a_k \leq M_{-r}(a) \leq q^{-1/r} a_k$$

Again, as $r \to \infty$, $-1/r \to 0$ and this proves the result.

These propositions inspire us to write

$$M_0(a) = G(a)$$

$$M_\infty(a) = \mathrm{Max}(a)$$

$$M_{-\infty}(a) = \mathrm{Min}(a)$$

We can then state our last result as

$$M_{-\infty}(a) < M_r(a) < M_\infty(a)$$

(for all finite r, unless all a are equal or $r \leq 0$ and some a is zero.)

We now prove an important proposition.

Proposition. ~~Cauchy's Inequality~~. $M_r(a) < M_{2r}(a)$ for $r>0$, unless all a are equal.

Proof. This relies on the following theorem:
(Cauchy's inequality)

$$(*) \quad \left(\sum ab\right)^2 < \sum a^2 \sum b^2, \text{ unless } (a),(b) \text{ proportional}$$

we can prove this by writing

$$\left(\sum a^2\right)\left(\sum b^2\right) - \left(\sum ab\right)^2 = \sum_{i,j} a_i^2 b_j^2 - \sum_{i,j} a_i a_j b_i b_j$$

$$= \sum_{i,j} a_i^2 b_j^2 - a_i b_i a_j b_j$$

$$= \sum_{i,j} a_i^2 b_j^2 - (a_i b_j)(a_j b_i)$$

Here's a sneaky trick; renaming $i \leftrightarrow j$ doesn't change the sum, so this is

$$= \frac{1}{2}\left(\sum_{i,j} a_i^2 b_j^2 - (a_ib_j)(a_jb_i) + a_j^2 b_i^2 - (a_jb_i)(a_ib_j) \right)$$

$$= \frac{1}{2}\left(\sum_{i,j} \left((a_ib_j)^2 - 2(a_ib_j)(a_jb_i) + (a_jb_i)^2\right) \right)$$

$$= \frac{1}{2} \sum_{i,j} (a_ib_j - a_jb_i)^2 \geq 0.$$

We have now proved our claim. But

$$M_r(a) = \left(\frac{\sum pa^r}{\sum p}\right)^{1/r}, \quad M_{2r}(a) = \left(\frac{\sum pa^{2r}}{\sum p}\right)^{1/2r}$$

so if we ~~square~~ raise both to the $2r$ power, we want to show

$$\frac{\left(\sum pa^r\right)^2}{\left(\sum p\right)^2} < \frac{\sum pa^{2r}}{\sum p}$$

or

$$\left(\sum pa^r\right)^2 < \sum p \sum pa^{2r}$$

If we let $(c) = \sqrt{p}$, $(d) = \sqrt{p}\, a^r$,
we see that $cd = pa^r$, $c^2 = p$, $d^2 = pa^{2r}$
and the desired inequality reduces to $(*)$. $\therefore$

We are now ready to prove a major result:

Theorem of the arithmetic and geometric means.

$$G(a) < U(a), \text{ unless all } a \text{ are equal.}$$

Proof. We know

$$U(a) = M_1(a) > M_{1/2}(a) > M_{1/4}(a) > \dots > M_0(a) = G(a),$$

using our proposition that $\lim_{r \to 0} M_r(a) = G(a)$.

This is the shortest proof, but not the most elementary. A cute proof (actually your book gives at least 5 proofs!) goes like this.

Suppose we assume that among all (a) with $U(a) =$ constant, there is some a^* which maximizes $G(a)$. We claim a^* has all a equal.

If the a are not all equal, let

$$a_1 = \text{Min } a \quad < \quad a_2 = \text{Max } a.$$

If we substitute

$$b_1 = \frac{a_1 + a_2}{2}, \quad b_2 = \frac{a_1 + a_2}{2}, \quad b_i = a_i$$

then $U(b) = U(a)$, but $G(b)$ contains

$$\left(\frac{a_1 + a_2}{2}\right)^2 \text{ instead of } a_1 a_2$$

and

$$\left(\frac{a_1 + a_2}{2}\right)^2 = \frac{a_1^2}{4} + \frac{a_1 a_2}{2} + \frac{a_2^2}{4}.$$

But

$$a_1^2 + a_2^2 - 2a_1 a_2 = (a_1 - a_2)^2 > 0, \text{ if } a_1 \neq a_2$$

so this is

$$> a_1 a_2.$$

Thus $G(b) > G(a)$. Hence, the max of $G(a)$

most occur when all (a_i) are equal.
But at that point,

$$G(a) = \prod a_1^{q_1} \cdots a^{q_n} = a^{\sum q_i} = a$$

and

$$U(a) = q_1 a_1 + \cdots + q_n a = \left(\sum q\right) a = a,$$

so G can never be greater than U.

Next time we'll consider Hölder's inequality!