~~Elementary~~ Inequalities

Many areas of mathematics are touched by the study of inequalities. In this class we'll focus on applications in geometry, but we will also cover analysis and number theory as appropriate.

In general, we are motivated by questions like:

> Of all curves in the plane of fixed length, which has the largest area?

Or

> A car has a turning radius of 20 ft. ~~The s~~ What is the shortest path for the car which starts at (0,0) heading in the (1,0) direction and ends at (10,10) heading in the (0,1) direction?

Or

Of all curves in the plane of fixed length, which has the largest average (cubed) chord length?

These questions are often quite difficult (in fact, the second was solved in the 1990's in its full generality ~~and~~ and the third is an open question), ~~and~~ and they are very different. But at some point proving answers to these (and many other) problems comes down to showing

$$f(x) \leq g(x)$$

over some range of x values.

The object of this class is to give you a (mostly classical) toolkit for approaching such questions in a systematic way. Learning these tools now will help you for the rest of your mathematical life.

A word about your book. Your book was written in the early 1930's, by 3 giants of analysis: Hardy and Littlewood are responsible for much of the framework of modern integration theory and Polya is known for

The book is written in an older, more advanced style which takes a bit of getting used to. We will have reading assignments, but I'll also post notes on the course webpage.

We'll have regular homework, one take-home exam, and a final project where you master and write up an application of the techniques from class.

Grades: 40% homework
30% midterm
30% final project

Let's begin!

Elementary Mean Values.

We will consider sets of non-negative

$$a_1, \ldots, a_n \quad \text{denoted } (a)$$

and some (nonzero) real r.

Definition. We say (a) is proportional to (b) if there are two numbers λ, μ not both zero so that

$$\lambda a_i = \mu b_i \quad \text{for } i \in 1, \ldots, n.$$

We let the null set be (a) with $a_i = 0$ and note that it is proportional to every (b).

Definition. The r-power mean of (a)

$$M_r(a) = \left(\frac{1}{n} \sum a^r \right)^{1/r}$$

unless $r = 0$ or ($r < 0$ and some $a_i = 0$).
In the second case, $M_r(a) = 0$ by definition.

These means are averages of the rth powers of numbers in (a).

Definition.

$$U(a) = M_1(a) = \underline{\text{arithmetic mean}}$$
or average

$$H(a) = M_{-1}(a) = \underline{\text{harmonic mean}}$$
$$= \left(\frac{n}{1/a_1 + \dots + 1/a_n} \right)$$

We also define

Definition. The <u>geometric mean</u> is given by

$$G(a) = \sqrt[n]{a_1 \cdots a_n} = \sqrt[n]{\prod a}$$
product

It is even more useful to consider <u>weighted means</u>. Suppose

$$p_i > 0 \quad \text{for } i \in 1, \dots, n$$

Definition. The rth power mean weighted by p is given by

$$M_r(a,p) = \left(\frac{\sum pa^r}{\sum p} \right)^{1/r}$$

Similarly,

$$G(a,p) = \left(a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \right)^{\frac{1}{\sum p}}$$

~~Example~~. These weighted means may look unfamilar, but you have seen these constructions before.

Example. If $(a = x_1, x_2, \ldots, x_n = b)$ is an equal-length subdivision of (a,b) and (a) is given by $a_i = f(x_i)$, then if $(p) = 1, 2, 2, \ldots, 2, 1$ we have

$$\int_a^b f\, dx \approx 2(n-1) M_1(a,p)$$

by the Trapezoid rule for numerical integration.

Homogeneity.

If $f(Kx) = K^d f(x)$, we say f is homogenous of degree ~~p~~d in x. An important special case is when $d=0$ and $f(Kx) = f(x)$.

Lemma. $M_r(a,p)$ and $G(a,p)$ are homogenous of degree 0 in (p).

Proof.

$$M_r(a,\lambda p) = \left(\frac{\sum \lambda p a^r}{\sum \lambda p} \right)^{1/r}$$

$$= \left(\frac{\lambda \sum p a^r}{\lambda \sum p} \right)^{1/r}$$

$$= \left(\frac{\sum p a^r}{\sum p} \right) = M_r(a,p).$$

$$G_{\text{g}}(a, \lambda p) = \left(a_1^{\lambda p_1} \cdots a_n^{\lambda p_n}\right)^{\frac{1}{\Sigma \lambda p}}$$

$$= \left(\left(a_1^{p_1} \cdots a_n^{p_n}\right)^{\lambda}\right)^{\frac{1}{\Sigma \lambda p}}$$

$$= \left(a_1^{p_1} \cdots a_n^{p_n}\right)^{\frac{\lambda}{\Sigma \lambda p}}$$

$$= \left(a_1^{p_1} \cdots a_n^{p_n}\right)^{\frac{1}{\Sigma p}} = G_{\text{g}}(a, p). \quad \therefore$$

Our theorems will work for weighted or unweighted means, so long as <u>two means being compared have the same weights.</u>

By the lemma, we may always assume

$$\Sigma p = \underline{1}.$$

If we are making this assumption, we'll call the weights q instead of p.

Your first homework assignment will checks that (and these are pretty easy)

1. $$M_r(a) = \left(U(a^r) \right)^{1/r}$$

2. $$G(a) = e^{U(\log a)} \qquad (a) > 0$$

3. $$M_{-r}(a) = \cancel{1}\, \frac{1}{M_r(1/a)} \qquad (a) > 0$$

4. $$M_{rs}(a) = \left(M_s(a^r) \right)^{1/r} \qquad (a) > 0 \text{ if } r,s < 0$$

5. $$U(a+b) = U(a) + U(b)$$

6. $$G(ab) = G(a)\,G(b)$$ (here $(ab) = a_1b_1, a_2b_2, \ldots, a_nb_n$)

7. $$M_r(\lambda a) = \lambda M_r(a)$$ (here λ is a scalar)

8. $$G(\lambda a) = \lambda G(a)$$

9. $M_r(a) \leq M_r(b)$ if $a_i \leq b_i$ for $i \in 1, \ldots, n$.

Next time: Inequalities relating the M_r.

Homework is posted on web page,
book available online (cheap! good!)
see you next time!