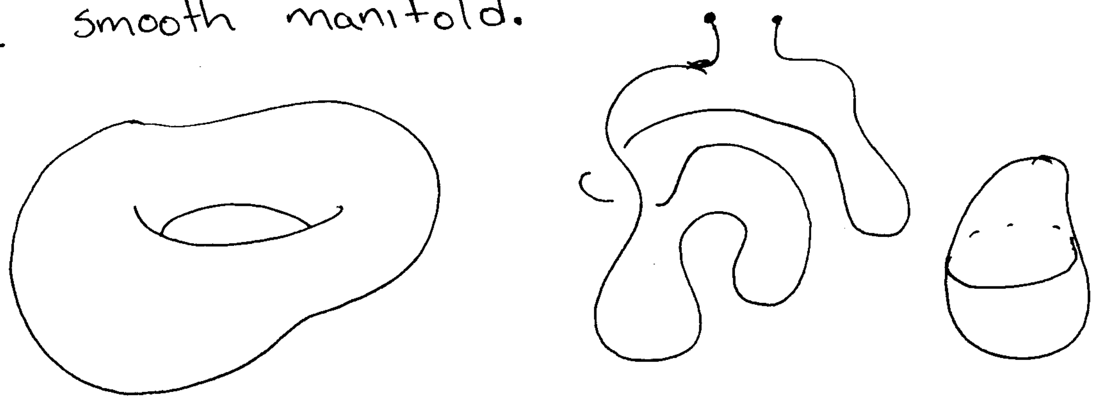
Introduction to Differential Topology

Overview of course.

The key idea of this course is that of the smooth manifold.



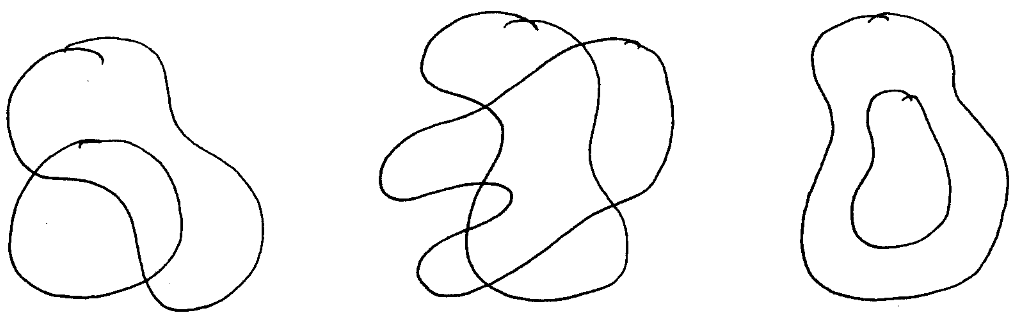
Locally, a smooth manifold is "identical" to an open set in $\mathbb{R}^n$. The class of smooth manifolds will be shown to be surprisingly ~~common~~ robust and useful. For example:

1) (Sard's Theorem.) Almost every level set of any smooth function is a smooth mfld.

2) (Transversality theorem.) Almost every intersection of ~~[illegible]~~ smooth manifolds is a smooth manifold.

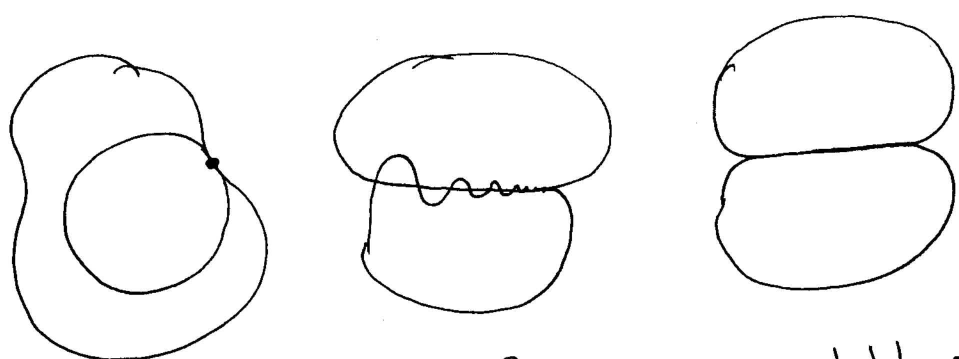
We will use this class of objects to develop a natural and useful idea: intersection theory.

Example. The intersection number of 2 smooth closed curves in the plane is even.



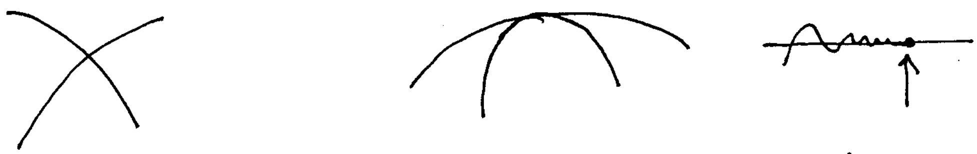
2   6   0

How could such a thing be proved? What does it even <u>mean</u>? How do we handle cases like



1? ∞?   uncountably many?

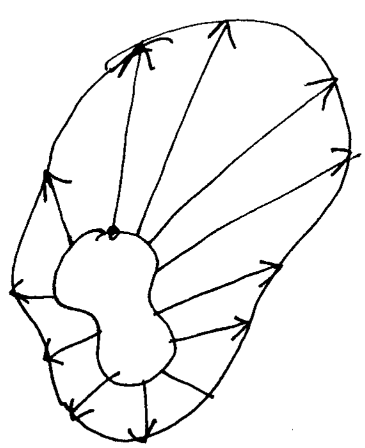
The key idea will be that of the <u>transverse</u> <u>intersection</u>. Roughly,



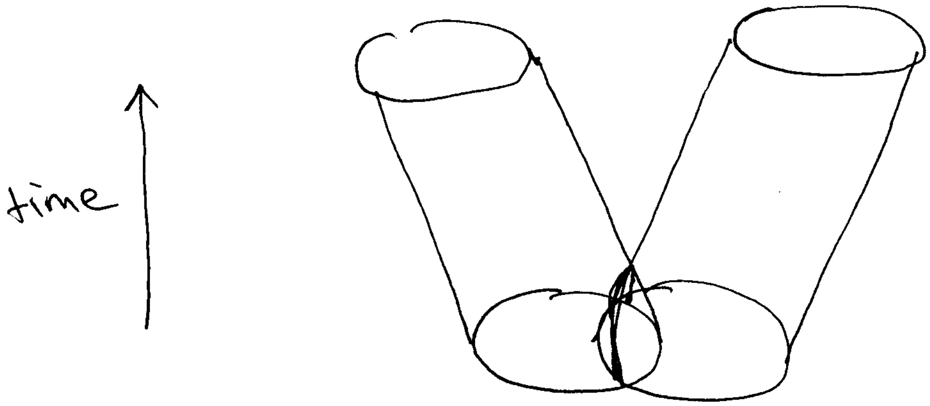
transverse   not transverse   not transverse

(we require that the tangent lines to our curves together span $\mathbb{R}^2$).

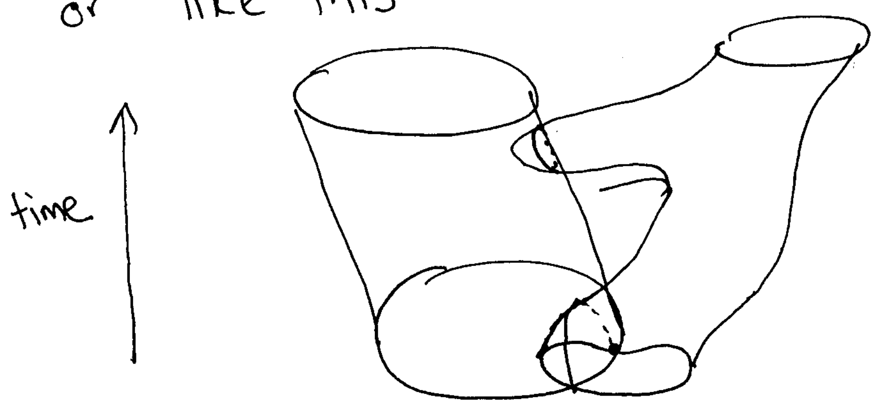
The <u>intersection</u> <u>number</u> counts <u>transverse</u> intersections only. We will prove that intersection number mod 2 is an invariant of smooth manifolds undergoing ~~continuous~~ smooth motions or <u>homotopies</u>.



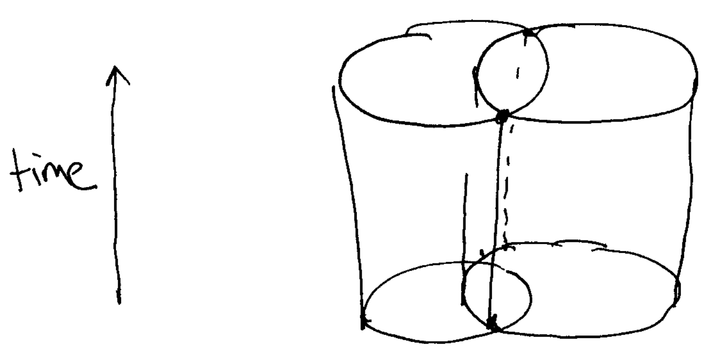
How? The pictures in intersection theory generally looks like this



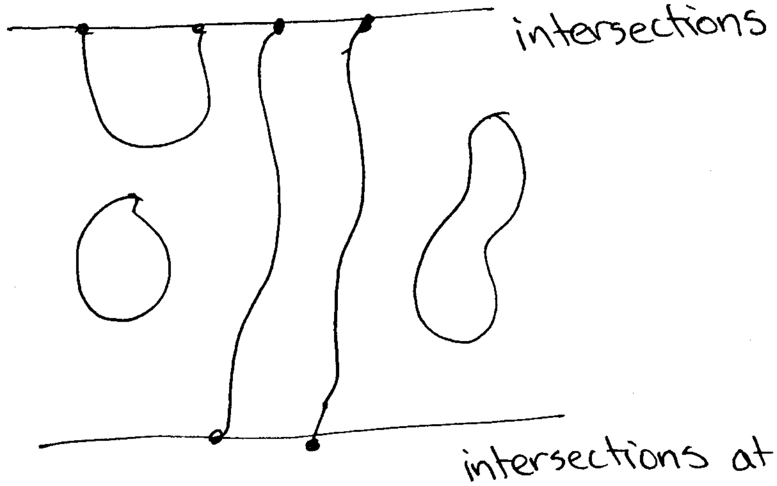
or like this



or like this

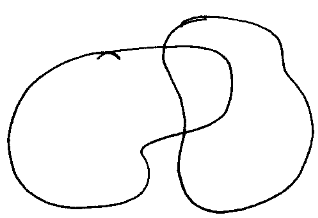


In each case, the surfaces swept out by the curves as they move are smooth 2-manifolds which intersect in some smooth 1-manifold



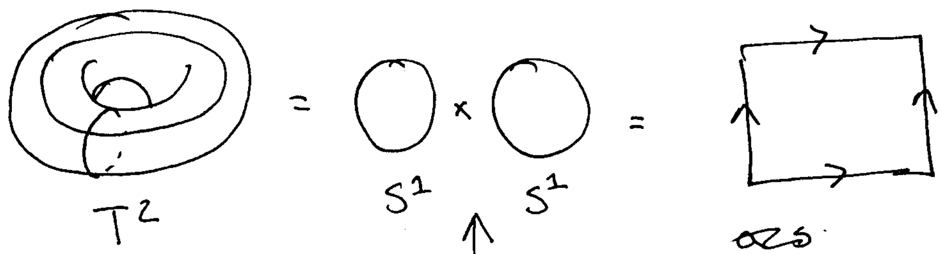
The intersection manifold has an <u>even</u> number of boundary points, so the intersection # is (mod 2) the same at start and finish.

Example. (again) The (mod 2) intersection number of two smooth closed plane curves is even.



Easy! By homotopy invariance, we need only find <u>a</u> configuration with transverse intersections and <u>count</u>.

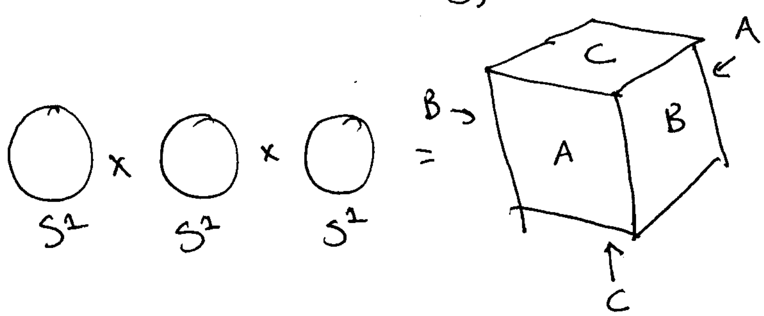
Let's consider a more complicated example, manifold. Recall the 2-torus:



$T^2 = S^1 \times S^1 =$ a square with identified edges.

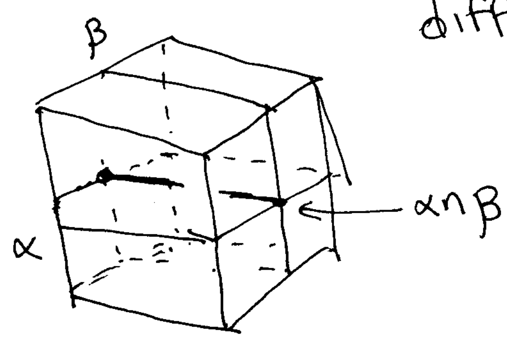
Proof: Every point on doughnut is specified by two independant angles.

We can, similarly, build a 3-torus T^3.



$S^1 \times S^1 \times S^1 =$ a cube with identified faces.

Compute the "intersection number" of two different ~~sub~~ T^2's inside T^3.



How? And in what group should this "number" live?

We expect the intersection of ~~2~~ two 2-mflds in a 3-mfld to be 1-dimensional.

~~To Compute~~

We will need to develop homology and cohomology theory to really answer this question.

These theories associate n groups

$$H_0(M), \dots, H_n(M)$$

to an n-manifold which count K-dimensional structures in M for $K \in 0, \dots, n$.

In this world the "cap product" of ~~α and β~~ $\alpha \in H_j(M)$ and $\beta \in H_K(M)$ will define a homotopy invariant "intersection class" ~~in~~

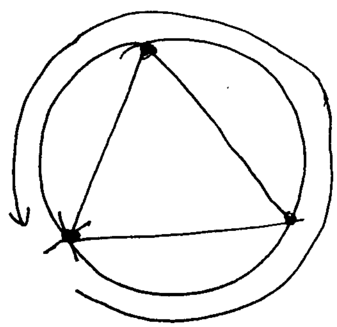
$$\alpha \cap \beta \in H_{j+K-n}(M).$$

Application. M^6 = triples of points in $\mathbb{R}^2$

α = triples of points in $\mathbb{R}^2$ forming equilateral triangles (4 dimensional)

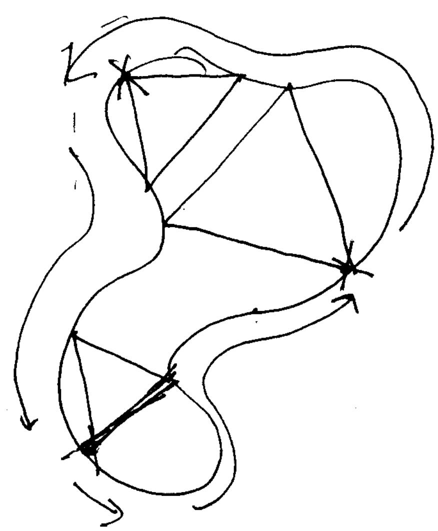
β = triples of point on a smooth closed curve γ.

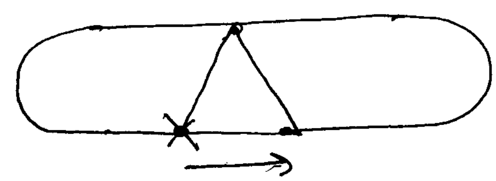
$\alpha \cap \beta$ = a loop moving one vertex around γ
~~which~~



we check it on circle.

And know it for <u>all</u> curves!





challenge for students:

Can you find the loop of equilateral triangles that moves x around this curve?